

FUNDAMENTAL FIELDS IN THE DEFORMED W-ALGEBRAS $\mathcal{W}_{q,t}(\mathfrak{g})$

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1. The Deformed W-Algebra $\mathbf{W}_{q,t}(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra and q, t generic parameters in $\mathbb{C}^\times$. In [FR98], Frenkel and Reshetikhin define the deformed W -algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ as follows.

The Heisenberg algebra $\mathbf{H}_{q,t}(\mathfrak{g})$

We define $\mathbf{H}_{q,t}(\mathfrak{g})$ as the polynomial ring generated by the fundamental weight variables $Y_i(z)$ for $i \in I$ and their shifts:

$$\mathbf{H}_{q,t}(\mathfrak{g}) = \mathbb{C}(q, t)[Y_i(za)^{\pm 1}]_{i \in I, a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$$

The simple root-type variables $A_i(z)$ are defined directly in terms of the fundamental weights $Y_j(z)$ via shifts encoding the quantum Cartan matrix:

$$\begin{aligned} : A_i(z) := & Y_i(zq^{-r_i t}) Y_i(zq^{r_i t^{-1}}) \prod_{C_{ji}=-1} Y_j(z)^{-1} \prod_{C_{ji}=-2} Y_j(zq^{-1})^{-1} Y_j(zq)^{-1} \times \\ & \times \prod_{C_{ji}=-3} Y_j(zq^{-2})^{-1} Y_j(z)^{-1} Y_j(zq^2)^{-1} : \end{aligned}$$

Screening Operators $S_i(z)$

We introduce the screening operators $S_i^\pm(z) = \sum_{m \in \mathbb{Z}} S_{i,m}^\pm z^{-m}$ which act on this ring. They are characterized by specific difference relations with the variables, satisfying equations such as:

$$: S_i^-(z) := q^{2r_i t^{-2}} : A_i(z t) S_i^-(z t^2) : .$$

$S_i^+(z)$ satisfies a similar relation with q^{-r_i} instead of t .

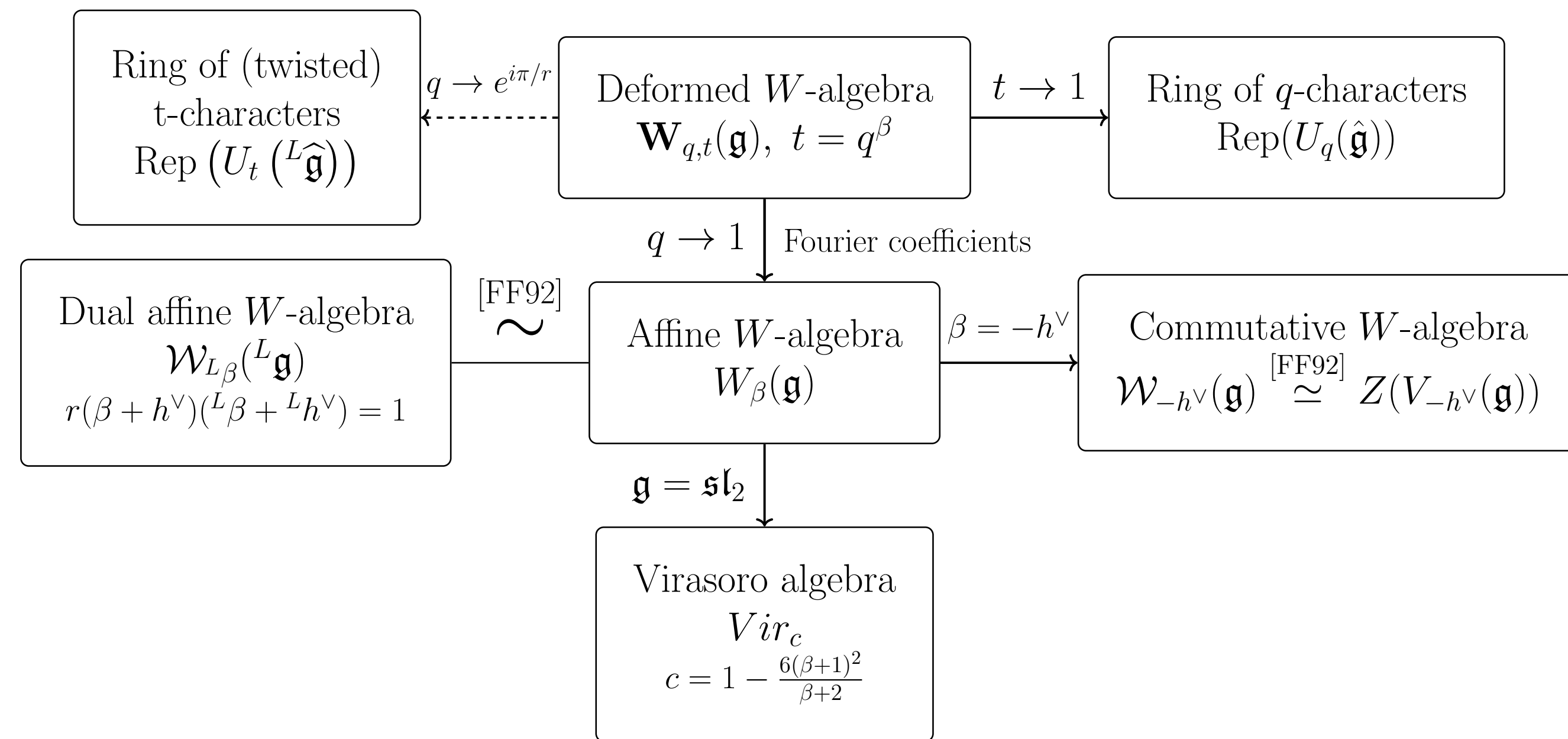
The Deformed W-Algebra $\mathbf{W}_{q,t}(\mathfrak{g})$

The deformed W -algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ is defined as the subspace of elements that commute with the screening charges, i.e., the intersection of the kernels of all screening operators:

$$\mathbf{W}_{q,t}(\mathfrak{g}) = \bigcap_{i \in I} \ker[S_{i,1}^\pm, \cdot]$$

2. Landscape of W -algebras

The theory of W -algebras bridges conformal field theory, representation theory, and quantum groups. The **deformed W -algebra** acts as a universal structure that specializes to various foundational algebras:



3. The Generation Algorithm

Inspired by the Frenkel-Mukhin algorithm [FM01] for the q -character theory of $U_q(\hat{\mathfrak{g}})$, where weights are computed by subtracting simple roots. We adapt this principle to systematically construct fields in $\mathbf{W}_{q,t}(\mathfrak{g})$.

Algorithm Steps:

- Assumptions:** We assume the initial monomial to be generic and to satisfy some condition on the spectral parameters (a monomial satisfying these conditions is called *regular*). If a monomial produced by the algorithm is not generic or not regular then we say that the algorithm *fails*.
- Initialization:** Let m_0 be a dominant generic regular monomial with coefficient 1.
- At each step:** For each produced monomial m , if a term $Y_i(za)$ of m has a positive power, then we "subtract" the simple root by constructing the new term $: m' := m A_i(z a q^{-r_i t})^{-1} :$. To cancel the residues, we define the coefficient as follows (the factors appear in [K22])

$$\lambda_{m'} = \lambda_m \prod_{b \neq a} \left(\frac{(1 - q^{-2r_i a b^{-1}})(1 - t^2 a b^{-1})}{(1 - a b^{-1})(1 - q^{-2r_i t^2 a b^{-1}})} \right)^{\deg_{(i,b)}(m)} .$$

Example (for $\mathfrak{g} = \mathfrak{sl}_2$):

Starting from a degree 2 monomial, the algorithm generates the following closure graph:

$$\begin{array}{ccc} & m^+ = : Y(z) Y(z q^{-2} t) : & \\ \swarrow & & \searrow \\ -\frac{q^2 t^{-1}}{q^2 - t} : Y(z q^{-2} t^2)^{-1} Y(z q^{-4} t) : & & -\frac{(q^4 - t)(t^3 - q^2)}{(q^4 - t^3)(q^2 - t)} : Y(z q^{-4} t^3)^{-1} Y(z) : \\ \searrow & & \swarrow \\ : Y(z q^{-2} t^2)^{-1} Y(z q^{-4} t^3)^{-1} : & & \end{array}$$

Theorem [A26]

The algorithm is well-defined. If it ends successfully, then the finite sum of generated terms produces a field that strictly lies within the deformed W -algebra $\mathbf{W}_{q,t}(\mathfrak{g})$.

There are two ways to apply this algorithm:

- Fix a dominant monomial satisfying the proper spectral conditions and apply the algorithm iteratively (e.g., computationally). This explicitly generates a field lying in $\mathbf{W}_{q,t}(\mathfrak{g})$.
- Based on our knowledge of q -character theory, we can intuit a set of monomials $\{m\}$ that obeys the combinatorial rules of the algorithm. The theorem then implies the existence of coefficients $\{\lambda_m\}$ such that $\sum \lambda_m m \in \mathbf{W}_{q,t}(\mathfrak{g})$.

5. Fundamental Fields & F-R Conjecture

A field whose leading term is $T_i(z) = Y_i(z) + \dots$ is called the **i -th fundamental field**.

Conjecture [FR98]

For each $i = 1, \dots, \ell$, there exists a field $T_i(z)$ in $\mathbf{W}_{q,t}(\mathfrak{g})$, such that $T_i(z) = Y_i(z) +$ the sum of elementary terms of the form:

$$c(q, t) : Y_i(z) A_{i_1}(z q^{a_1} t^{b_1})^{-1} \dots A_{i_k}(z q^{a_k} t^{b_k})^{-1} :$$

(where $c(q, 1)$ is a positive integer independent of q).

The set of weights of these terms, counted with multiplicity $c(q, 1)$, is exactly the set of weights of the finite-dimensional irreducible representation V_{ω_i} of $U_q(\hat{\mathfrak{g}})$ with highest weight ω_i .

This conjecture provides a profound bridge between the structure of deformed W -algebras and the q -character theory of quantum affine algebras.

6. Algorithm Results & New Conjectures

We applied our algorithm to systematically construct these fundamental fields. While the first method described in Section 4 only allowed computations for exceptional types, the second handles the classical Lie types.

Lie Type $\mathfrak{g}$	Algorithm Success for $T_i(z)$
$A_\ell, B_\ell, C_\ell, G_2$	For all nodes i
D_ℓ, F_4	For extremal nodes i
E_6	For $i = 1, 5$
E_7	For $i = 6$
E_8	Never

Remark: The specific cases where the algorithm successfully computes the fundamental field correspond exactly to the **thin fundamental representations** of $U_q(\hat{\mathfrak{g}})$ (representations where all weight spaces have dimension 1).

Conjectures on Convergence

Based on these results, we formulate two conjectures:

- If the specialization $t = 1$ of m_0 is the unique dominant monomial of a q -character of a **thin** representation $V \in \text{Rep}(U_q(\hat{\mathfrak{g}}))$, then the algorithm starting from m_0 converges.
- Let m^+ be a dominant monomial in $\mathbf{H}_{q,t}(\mathfrak{g})$. The algorithm starting from m^+ works **if and only if** there exists a field without derivative terms that admits m^+ as its unique dominant monomial.

References

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