FUNDAMENTAL FIELDS IN THE DEFORMED W-ALGEBRAS

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra. Frenkel and Reshetikhin introduced the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ in [FR98]. In this work, we propose a formal reformulation of this definition in a slightly different context. In this framework, we introduce an explicit algorithm inspired by the Frenkel-Mukhin algorithm [FM01] that produces elements of the deformed W-algebra starting from a given dominant monomial m satisfying some degree conditions. Then, we apply this algorithm to construct explicitly some specific elements of $\mathbf{W}_{q,t}(\mathfrak{g})$. In particular, we apply this to prove a conjecture of Frenkel and Reshetikhin in [FR98] in types B_ℓ, C_ℓ, and for some nodes in other types. This framework opens up new possibilities for studying explicitly fields in the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$.

Contents

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1. Introduction

A brief historical review of W-algebras. The study of W-algebras and their deformations has been a central theme at the intersection of conformal field theory, integrable systems, and representation theory. Let us present a quick review of their story (see [BS93] for a more complete review).

The first W-algebra is introduced by Zamolodchikov in [Zam85] as an extension of the Virasoro algebra. Then, Fateev and Lukyanov generalize this construction to $\mathfrak{sl}_N$ and define the W_N-algebras [FL88]. In [FF90], Feigin and Frenkel prove that this W_N-algebra can be obtained by a Drinfeld-Sokolov reduction of the affine algebra $\widehat{\mathfrak{sl}_n}$ with respect to a principal nilpotent element of $\mathfrak{g}$. They derive explicitly this W_N-algebra as a BRST-cohomology algebra. For all simple Lie algebra $\mathfrak{g}$, the quantum Drinfeld-Sokolov reduction of $\widehat{\mathfrak{g}}$ at level k gives a generalization of the W_N-algebra denoted $W_k(\mathfrak{g})$. At the critical level $k = -h^\vee$, the W-algebra becomes the center of the universal affine vertex algebra $V_{-h^\vee}(\mathfrak{g})$ at the critical level [Fre91]. Moreover, the affine W-algebras $W_k(\mathfrak{g})$ present a remarkable duality. If ${}^L\mathfrak{g}$ is the Langlands dual of $\mathfrak{g}$, then $W_k(\mathfrak{g})$ and $W_{k'}({}^L\mathfrak{g})$ are isomorphic if $r(\beta + h^\vee)({}^L\beta + {}^Lh^\vee) = 1$, where r is the maximal number of edges connecting two vertices in the Dynkin diagram of $\mathfrak{g}$ ([FF92]). Later on, Awata, Kubo, Odake and Shiraishi defined a two-parameters deformation of the Virasoro algebra [SKAO96] called q-Virasoro. Then the same authors define in parallel to Feigin and Frenkel the $q - W_N$-algebras [AKOS96, FF96] which are deformations of the W_N-algebras. Finally, Frenkel and Reshetikhin extend this definition and define the two parameters deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ associated to a simple Lie algebra $\mathfrak{g}$ in [FR98]. These deformed W-algebras are defined as the intersection of the kernels of screening operators acting on a double deformed Heisenberg algebra $\mathbf{H}_{q,t}(\mathfrak{g})$. In [FR98], Frenkel and Reshetikhin highlight that the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ is remarkably connected with the analytic Bethe Ansatz in integrable models associated with the quantum affine algebras $U_q(\widehat{\mathfrak{g}})$, $U_t({}^L\widehat{\mathfrak{g}})$ and $U_t({}^L\widehat{({}^L\mathfrak{g})})$, where $\widehat{\mathfrak{g}}$ denotes the corresponding affine Kac-Moody algebra, ${}^L\mathfrak{g}$ is the Langlands dual of $\mathfrak{g}$, and $U_q(\widehat{\mathfrak{g}})$ is the quantum affine algebra associated to $\mathfrak{g}$ defined by Drinfeld and Jimbo in [Dri85, Jim85].

In [KP18b, KP18a], Kimura and Pestun provide a quiver gauge theoretic construction of Frenkel and Reshetikhin deformed W-algebras, providing a geometric point of view on the deformed W-algebras. In [Neg22], Neguţ defines another deformed W-algebra in type A, that deforms the W-algebra of $\mathfrak{gl}_{nr}$ with respect to a rectangular nilpotent. This definition generalizes the definition of Frenkel and Reshetikhin in type A. The deformed W-algebra is also studied in [AFO18, AFR20, FJM25, HKc22, Koj22, LNZ18, NZ20, Sev02, Tan18].

While the original construction provided a powerful framework for understanding quantum integrable systems, the precise algebraic nature of these algebras and their behavior under classical limits presents significant technical challenges. In this document, we define a formal context to this deformed W-algebra, providing new tools to explicitly compute its elements.

The deformed W-algebra in a formal context. Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ. Let $I = [\![1, \ell]\!]$. In this work, we aim to set the deformed W-algebras $\mathbf{W}_{q,t}(\mathfrak{g})$ defined in [FR98] in a new formal setting. We work in the ring of formal power series $K = \mathbb{C}[[h, \beta]]$ and we fix $q = e^h$, and $t = e^{h\beta}$ two parameters. We recall the definitions of the double deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ and the screening operators in [FR98] adapted in our context. Then, $\mathbf{H}_{q,t}(\mathfrak{g})$ denotes the vector space generated by monomials in (respectively fundamental weight-type and simple root-type) variables $Y_i(za), A_i(za) \in \mathcal{H}_{q,t}(\mathfrak{g})[\![z^{\pm 1}]\!]$, $i \in I$, $a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$ with respect to the normally ordered product. Finally, the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ denotes the vector subspace of $\mathbf{H}_{q,t}(\mathfrak{g})$ generated by the fields commuting with screening operators $S_i^{\pm}$.
If $\omega_1, \ldots, \omega_\ell$ are the fundamental weights of the simple Lie algebra $\mathfrak{g}$, then there exist *fundamental* representations V_{ω_i} of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ (see [CP94] for more details). In [FR98], Frenkel and Reshetikhin formulate the following conjecture :

Conjecture 1 ([FR98], Conjecture 1)**.** *For each $i = 1, \ldots, \ell$, there exists a field $T_i(z)$ in $\mathbf{W}_{q,t}(\mathfrak{g})$, such that $T_i(z) = Y_i(z) +$ the sum of elementary terms of the form*

$$c(q,t) : Y_i(z) A_{i_1}(zq^{a_1} t^{b_1})^{-1} \cdots A_{i_k}(zq^{a_k} t^{b_k})^{-1} :$$

(where $c(q, 1)$ is a positive integer independent of q). Furthermore, the set of weights of these terms counted with multiplicity $c(q, 1)$ is the set of weights of the finite-dimensional irreducible representation V_{ω_i} of $U_q(\widehat{\mathfrak{g}})$ with highest weight ω_i, where the weight of

$$: \prod_{j=1}^{d} Y_{i_j}(za_j)^{\varepsilon_j} :$$

is

$$\sum_{j=1}^{d} \varepsilon_j \omega_{i_j}.$$

This would prove a strong link between the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ and the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. This conjecture is proved for $\mathfrak{g}$ of type A_ℓ ([FF96, AKOS96, FR98]). Frenkel and Reshetikhin proved this conjecture in all classical types for $i = 1$ in [FR98]. Bouwknegt and Pilch proved this conjecture for all simple Lie algebra $\mathfrak{g}$ of rank 2 (Appendix A in [BP98]).
However, as far as we know this conjecture has not been proved yet for the other classical types because of the lack of tools to compute systematically the fields in $\mathbf{W}_{q,t}(\mathfrak{g})$.
We draw attention to the fact that the elements of this deformed W-algebra correspond to Nekrasov's qq-characters [Nek16, KP18b]. These elements must be distinguished from Nakajima's (q,t)-characters [Nak01, Her04] and from the interpolating (q,t)-characters of Frenkel and Hernandez [FH11]. The latter are commutative algebraic polynomials defined to emulate the behavior of the fields in $\mathbf{W}_{q,t}(\mathfrak{g})$.

An algorithm to compute explicitely fields in $\mathbf{W}_{q,t}(\mathfrak{g})$. In this document, the reformulation of the definition of $\mathbf{W}_{q,t}(\mathfrak{g})$ allows us to construct explicitely its elements using an algorithm inspired by the Frenkel-Mukhin algorithm [FM01] for computing q-characters.

Our algorithm is similar to the Frenkel-Mukhin algorithm in the fact that it works step by step, starting from a *dominant* monomial (that is a monomial in $(Y_i(za))_{i\in[\![1,\ell]\!],a\in\mathbb{C}^*q^{\mathbb{Z}}t^{\mathbb{Z}}}$), and at each step, multiplying it by variables $A_i(za)^{-1}$ until we get an *antidominant* monomial in $(Y_i(za)^{-1})_{i,a}$.
However, the intricated structure of the double deformed Heisenberg algebra implies that unlike the set of q-characters, the deformed W-algebra is not a ring, and we cannot give a complete characterization of the fields in a fixed $\mathfrak{sl}_2$-direction. Thus, our algorithm differs from the Frenkel-Mukhin algorithm. The first difference is the fact that at each step and for each monomial, we have to isolate each *admissible* variable $Y_i(za)$, multiply the monomial by $A_i(zaq^{-r_i}t)^{-1}$, and then define a coefficient for the new monomial.
The second and main difference is the fact that the coefficients are not defined as maxima as in [FM01], but they are defined as residues of rational functions in $q^{\pm 1}, t^{\pm 1}$.
We prove the following theorem which makes this algorithm a key tool to compute elements in $\mathbf{W}_{q,t}(\mathfrak{g})$:
Theorem 1. *The algorithm is well-defined. Moreover, if it does not fail and ends in finitely many steps, then it gives a field $T(z) \in \mathbf{W}_{q,t}(\mathfrak{g})$.*

As an application, we prove Conjecture 1 in new cases:

Theorem 4. *Conjecture 1 holds in types A_ℓ (for all $i \in I$), B_ℓ (for all $i \in I$), C_ℓ (for all $i \in I$), D_ℓ (for $i = 1, \ell-1, \ell$), E_6 (for $i = 1, 5$), E_7 (for $i = 6$), F_4 (for $i = 1, 4$) and G_2 (for $i = 1, 2$).*

This algorithm provides a powerful tool to construct explicitly fields in $\mathbf{W}_{q,t}(\mathfrak{g})$, however, in general, the explicit computation of the fields for $\mathfrak{g}$ of classical type presents additional difficulties. This will be studied in an upcoming paper.
Finally, we conjecture that for any dominant monomial m, our algorithm terminates in finitely many steps without failing if and only if there exists a generic field in $\mathbf{W}_{q,t}(\mathfrak{g})$ having m as its unique dominant monomial. Furthermore, the following conjecture highlights a potential connection between the fields generated by this algorithm and the thinness of the corresponding representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.

Conjecture 3. *Our algorithm generates precisely those fields of $\mathbf{W}_{q,t}(\mathfrak{g})$ whose unique dominant monomial, when specialized to $t = 1$, coincides with the dominant monomial of a q-character of a special thin representation of $U_q(\widehat{\mathfrak{g}})$. Here, a special thin representation is defined as one whose q-character possesses a single dominant monomial and has all its coefficients equal to 1.*

Theorem 4 proves this for fundamental fields.

Structure of the paper. This article is organized as follows:
In Section 2, we recall the definition of the double deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ introduced by Frenkel and Reshetikhin in [FR98], its Fock representation in our context, and the completion of $\mathcal{H}_{q,t}(\mathfrak{g})$. We prove that the Fock representation is faithful.

In Section 3, we recall the definition of the screening currents and of the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ as the intersection of the kernels of the screening operators acting on $\mathbf{H}_{q,t}(\mathfrak{g})$. We prove that $\mathbf{H}_{q,t}(\mathfrak{g})$ is isomorphic to a polynomial algebra $K[Y_{i,a}^{\pm 1}]$, where $Y_{i,a}$ is identified with $Y_i(za)$ and where the product is the normally ordered product. Then, we compute commutation relations of the screening currents with the generators of $\mathbf{H}_{q,t}(\mathfrak{g})$, and discuss the coefficients of the fields in $\mathbf{W}_{q,t}(\mathfrak{g})$.

In Section 4, we present the central algorithmic result, including a graph-based representation of the algorithm inspired by [FR99], and explicit examples of fields in $\mathbf{W}_{q,t}(\mathfrak{g})$. These results are the first main results of this article.

In Section 5, we apply our algorithmic results to prove Conjecture 1 in types B_ℓ, C_ℓ, and in other new cases. This proof is the second main result of this article.

Finally, in Section 6, we formulate conjectures on the behavior of our algorithm and on a possible link between the fields it produces and the thinness of the corresponding representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.

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2. The double deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$

In this section, we recall the definition of the double deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$. This definition is introduced by Frenkel and Reshetikhin for a simple Lie algebra $\mathfrak{g}$ of any type in [FR98]. In type A_ℓ, the definition appears in [AJMP96, AKOS96, FF96]. Here we give the same definition however to justify the exponential and logarithm, we define the parameters q, t as formal power series in h.

Let h, β be generic variables. Let $(q,t) = (e^h, e^{h\beta}) \in \mathbb{C}[[h,\beta]]^2$. These are formal variables. In all this document, h can be specialized in $\mathbb{C}\backslash i\pi\mathbb{Q}$ and we can write $t = q^\beta$. Moreover, if $\gamma \in \mathbb{C}$, then $q^\gamma := e^{h\gamma}$ and $t^\gamma := e^{h\beta\gamma}$ are well defined as formal variables. We can now work on the ring $K = \mathbb{C}[[h,\beta]]$.

Notation 1. For all $n \in \mathbb{Z}$, we set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Remark 1. Note that $[n]_q \in \mathbb{C}[[h]]$ and $[n]_q \in n + h\mathbb{C}[\![h]\!]$.

Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ. Let $I = \{1, \dots, \ell\}$. Let $(\cdot,\cdot)$ be the invariant inner product on $\mathfrak{g}$ normalized such that for all maximal root α, $(\alpha,\alpha) = 2$ (see [Kac90]). Let $\{\alpha_1, \dots, \alpha_\ell\}$ and $\{\omega_1, \dots, \omega_\ell\}$ be the sets of simple roots and of fundamental weights of $\mathfrak{g}$, respectively. We have:

$$(\alpha_i, \omega_j) = \frac{(\alpha_i,\alpha_i)}{2}\delta_{i,j}.$$

Let r be the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$. Thus, $r = 1$ for simply-laced $\mathfrak{g}$, $r = 2$ for B_ℓ, C_ℓ, F_4, and $r = 3$ for G_2. Set

$$D = \operatorname{diag}(r_1, \dots, r_\ell),$$

where

$$r_i = r\frac{(\alpha_i,\alpha_i)}{2}. \tag{1}$$

All r_i's are integers; for simply-laced $\mathfrak{g}$, D is the identity matrix.
Now let $C = (C_{ij})_{1\le i,j\le \ell}$ be the *Cartan matrix* of $\mathfrak{g}$. Denote by $I = 2I_n - C$ the *incidence matrix*, and $B = (B_{ij})_{1\le i,j\le\ell} = DC$ be the symmetrized Cartan matrix:

$$B_{ij} = r(\alpha_i, \alpha_j).$$

In [FR98], the authors define $\ell \times \ell$ matrices $C(q,t)$, $D(q,t)$, and $B(q,t)$ with coefficients in K by the formulas

$$C_{ij}(q,t) = (q^{r_i}t^{-1} + q^{-r_i}t)\delta_{i,j} - [I_{ij}]_q, \tag{2}$$
$$D(q,t) = \mathrm{diag}([r_1]_q, \dots, [r_\ell]_q), \tag{3}$$
$$B(q,t) = D(q,t)C(q,t).$$

Thus,

$$B_{ij}(q,t) = [r_i]_q \left((q^{r_i}t^{-1} + q^{-r_i}t)\delta_{i,j} - [I_{ij}]_q\right). \tag{4}$$

It is easy to see that the matrix $B(q,t)$ is symmetric. For simply-laced $\mathfrak{g}$,

$$C_{ij}(q,t) = B_{ij}(q,t) = (qt^{-1} + q^{-1}t)\delta_{i,j} - I_{ij}.$$

Lemma 1. For all $n \geq 0$, the matrix $C(q^n,t^n) \in \mathrm{Mat}_{\ell\times\ell}(K)$ is invertible and its inverse lies in $\mathrm{Mat}_{\ell\times\ell}(K)$.

Proof. Let $n \in \mathbb{N}$. We consider $d = \det(C(q^n,t^n)) \in K$. If $d = 0$ then its image in the quotient $\overline{d} \in K/h$ is null. However, $\overline{d} = \overline{\det C}$. As $\mathfrak{g}$ is a simple Lie algebra, the Cartan matrix C is invertible. Thus $\det C \in \mathbb{C}\backslash\{0\}$ and $\overline{d} \neq 0$. Thus $C(q^n,t^n)$ is invertible. Moreover, $d = d_0 + f(h,\beta)$ where $f(h,\beta)$ is a formal power series in h,β with no constant terms and $d_0 \in \mathbb{C}^*$. Hence, d is invertible in $\mathbb{C}[[h,\beta]]$, and

$$C(q^n,t^n)^{-1} = \frac{1}{\det C(q^n,t^n)}\mathrm{Com}(C(q^n,t^n))^T \in \mathrm{Mat}_{\ell\times\ell}(K).$$

□

2.1. **Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$.** In this subsection, we present the definition of the double deformed Heisenberg algebra as introduced in [FR98]. This definition appears in type A_ℓ in [SKAO96].

Definition 1. Let $\mathcal{H}_{q,t}(\mathfrak{g})$ be the (double-deformed Heisenberg) algebra over the ring K with generators $x_i[n]$, $q^{\xi a_i[0]}$ and $e^{\gamma Q_i}$, with $i \in I$, $\xi \in \mathbb{C}[\beta]$, $\gamma \in \beta^{-1}\mathbb{C}[\beta]$, $n \in \mathbb{Z}\backslash\{0\}$ and relations

$$[x_i[n], x_j[m]] = n\frac{t^n - t^{-n}}{q^n - q^{-n}}B_{ij}(q^n,t^n), \delta_{n,-m} \tag{5}$$

$$[x_i[n], e^{\gamma Q_j}] = 0, \quad n \neq 0,$$
$$[e^{\gamma Q_j}, e^{\gamma' Q_{j'}}] = 0, \quad \gamma,\gamma' \in \beta^{-1}\mathbb{C}[\beta], j,j' \in I,$$
$$[q^{\xi a_i[0]}, e^{\gamma Q_j}] = (q^{\xi\gamma\beta B_{ij}} - 1)e^{\gamma Q_j}q^{\xi a_i[0]},$$
$$[q^{\xi a_i[0]}, x] = 0, \qquad x \in \langle x_j[n]\rangle_{j\in I, n<0}.$$

The $e^{\gamma Q_j}$ are called the *shift generators.*

We construct for all $i \in I$, $m \in \mathbb{Z}\backslash\{0\}$,

$$a_i[m] := \frac{q^m - q^{-m}}{n}x_i[m].$$

It implies the following commutation relation :

$$[a_i[n], a_j[m]] = \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n})B_{ij}(q^n,t^n).\delta_{n,-m}$$

Remark 2. The $(a_i[m])_{i\in I, m\neq 0}$ are the generators of the Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ defined in [FR98]. Here, because we consider the ring $\mathbb{C}[\![h,\beta]\!]$ we slightly change the definition and the Frenkel and Reshetikhin's Heisenberg algebra is a strict subalgebra of the Heisenberg algebra of this paper. If we work in the field $\mathbb{C}((h,\beta))$ instead of $\mathbb{C}[[h,\beta]]$, then both are equal.

The generators $a_i[n]$ are "simple root"-type elements of $\mathcal{H}_{q,t}(\mathfrak{g})$. As the deformed Cartan matrix is invertible, there is a unique set of "fundamental weight"-type element, $y_i[n], q^{\xi y_i[0]}, t^{\xi y_i[0]}$ $i = 1,\dots,\ell; n \in \mathbb{Z}\backslash\{0\}$, $\xi \in \mathbb{C}$ satisfying:

$$\forall 1 \leq j \leq \ell, \qquad a_j[n] = \sum_{i=1}^{\ell} C_{ij}(q^n, t^n) y_i[n], \qquad q^{\xi a_j[0]} = \prod_{i=1}^{\ell} q^{\xi C_{ij} y_i[0]}, \tag{6}$$

$$[a_i[n], y_j[m]] = \frac{1}{n}(q^{r_i n} - q^{-r_i n})(t^n - t^{-n})\delta_{i,j}\delta_{n,-m}. \tag{7}$$

We also put the following relations :

$$\forall \xi, xi' \in \mathbb{C}, \forall j, j' \in I, \qquad q^{\xi a_j[0]} q^{\xi' a_{j'}[0]} = q^{(\xi+\xi') a_{j'}[0]}, \qquad q^{0.a_j[0]} = e^{0.Q_j} = 1,$$

and same for $y_j[0]$.
They satisfy the following commutation relations:

$$[y_i[n], y_j[m]] = \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n}) M_{ij}(q^n, t^n)\delta_{n,-m}, \tag{8}$$

where $(M_{ij}(q,t))_{1\leq i,j\leq \ell}$ is the matrix $M(q,t) = D(q,t)C(q,t)^{-1}$.

2.2. **Fock representation π_μ of $\mathcal{H}_{q,t}(\mathfrak{g})$.** In this section we present the construction of the Fock representations of the double deformed Heisenberg algebra. This representation is introduced in [FR98, BP98]. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. We define

$$P := \mathbb{C}[\beta] \otimes_{\mathbb{C}} \mathfrak{h}^* = \{\sum_{i=1}^{k} \gamma_i \mu_i \ ; \ \gamma_i \in \mathbb{C}[\beta], \ \mu_i \in \mathfrak{h}^*\}.$$

Let us define the following subalgebras of $\mathcal{H}_{q,t}(\mathfrak{g})$

$$H^- := \langle x_i[n] \rangle_{i\in I, n<0}, \quad H^+ := \langle x_i[n], q^{\xi a_i[0]} \rangle_{i\in I, \xi\in\mathbb{C}[\beta], n>0},$$
$$\text{and} \quad H := \langle x_i[n], q^{\xi a_i[0]} \rangle_{i\in I, \xi\in\mathbb{C}[\beta], n\neq 0}$$

Let $\mu = \sum_{i=1}^{k} \gamma_i \mu_i \in P$, with $\gamma_i \in \mathbb{C}[\beta]$, $\mu_i \in \mathfrak{h}^*$. Let $\mathbb{C}_\mu$ be a one-dimensional vector space generated by a vector v_μ. We construct a structure of H^+-module on C_μ with the following actions :

$$\forall j \in I, n > 0 \qquad x_j[n] v_\mu = 0,$$
$$\forall j \in I \qquad q^{\xi a_j[0]} v_\mu = q^{\xi(\mu,\alpha_j)} v_\mu,$$

with $(\mu,\alpha_j) := \sum_{i=1}^{k} \gamma_i(\mu_i,\alpha_j)$. We can now define the Fock representation π_μ as follows :

$$\pi_\mu := \mathrm{Ind}_{H^+}^{H} \mathbb{C}^\mu = H^- \otimes_{H^+} \mathbb{C}_\mu,$$

The set π_μ is a representation of H and has a Poincaré-Birkhoff-Witt basis:

$$(x_{i_1}[n_1]x_{i_2}[n_2]\dots x_{i_m}[n_m]v_\mu)_{n_1\leq\dots\leq n_m<0}$$

Thus, the direct sum $\bigoplus_{\mu\in P}\pi_\mu$ is a representation of H. To extend it as a representation of $\mathcal{H}_{q,t}(\mathfrak{g})$ we have to define how the shift generators act on the elements of each π_μ. The shift generators commute with $(x_i[n])_{n\neq 0}$. Hence it is sufficient to define how it acts on each v_μ. Let $\mu\in P$, $\gamma\in\beta^{-1}\mathbb{C}[\beta]$, and $i\in I$. Thus $\mu+\gamma\beta r\alpha_i\in P$ and we define

$$e^{\gamma Q_i}\cdot v_\mu := v_{\mu+\gamma\beta r\alpha_i}.$$

The following proposition proves that this definition is compatible with the commutation relations of $\mathcal{H}_{q,t}(\mathfrak{g})$ and that it gives a well-defined faithful representation of $\mathcal{H}_{q,t}(\mathfrak{g})$. This may be well-known to experts, but we write a proof for completeness.

Proposition 1. *The vector space $\bigoplus_{\mu\in P}\pi_\mu$ is a well-defined, faithful representation of $\mathcal{H}_{q,t}(\mathfrak{g})$.*

Proof. To prove that it is a well-defined representation, it is sufficient to verify that this definition is compatible with the commutation relations between $e^{\gamma Q_i}$ and $q^{\xi a_i[0]}, t^{\xi a_i[0]}$. Let $\xi\in\mathbb{C}[\beta]$, $\mu\in P$, $\gamma\in\beta^{-1}\mathbb{C}[\beta]$, and $i,j\in I$

$$\begin{aligned} e^{\gamma Q_i}\cdot(q^{\xi a_j[0]}v_\mu) &= e^{\gamma Q_i}q^{\xi(\mu,\alpha_j)}v_\mu \\ &= q^{\xi(\mu,\alpha_j)}e^{\gamma Q_i}v_\mu \\ &= q^{\xi(\mu,\alpha_j)}v_{\mu+\gamma\beta r\alpha_i} \end{aligned}$$

and

$$\begin{aligned} q^{\xi a_j[0]}e^{\gamma Q_i}\cdot v_\mu &= q^{\xi a_j[0]}v_{\mu+\gamma\beta r\alpha_i} \\ &= q^{\xi(\mu+\gamma\beta r\alpha_i,\alpha_j)}v_{\mu+\gamma\beta r\alpha_i} \\ &= q^{\xi\gamma\beta r(\alpha_i,\alpha_j)}q^{\xi(\mu,\alpha_j)}v_{\mu+\gamma\beta r\alpha_i} \\ &= q^{\xi\gamma\beta B_{i,j}}q^{\xi(\mu,\alpha_j)}v_{\mu+\gamma\beta r\alpha_i} \end{aligned}$$

Hence,

$$[q^{\xi a_j[0]},e^{\gamma Q_i}]\cdot v_\mu = q^{\xi a_j[0]}e^{\gamma Q_i}\cdot v_\mu - e^{\gamma Q_i}\cdot(q^{\xi a_j[0]}v_\mu)$$

Hence, there is an algebra homomorphism

$$\rho:\mathcal{H}_{q,t}(\mathfrak{g})\longrightarrow End\left(\bigoplus_{\mu\in P}\pi_\mu\right)$$

sending the generators to the associated action on $\bigoplus_{\mu\in P}\pi_\mu$. Let us prove that ρ is injective.
Let $X=\sum_{i=1}^p Q_iP_iX_i$ be an element of $\mathcal{H}_{q,t}(\mathfrak{g})$ such that $\rho(X)=0$, with for all $i\in[\![1,p]\!]$,

$$P_i\in K[q^{\xi a_j[0]}]_{\xi\in\mathbb{C}[\beta],j\in I},\qquad Q_i\in K[e^{\gamma Q_j}]_{\gamma\in\beta^{-1}\mathbb{C}[\beta],j\in I},$$

and

$$X_i=X_i^-X_i^+,$$

where X_i^- (resp. X_i^+) is a monomial in the $x_j[n]$ with $n<0$ (resp. $n>0$). Without loss of generality, we assume that the Q_i are unitary monomials in the shift operators and that the couples (Q_i,X_i) are pairwise distinct.

Let $\alpha \in P$. For all i, let
$$v_{\mu_i} = Q_i \cdot v_\alpha, \quad \text{and} \quad p_i v_\alpha = P_i \cdot v_\alpha.$$
Let us prove that $p_i = 0$ for all $i \in [\![1, p]\!]$. Let us proceed by induction on the number of positive modes of X_i. For all $i \in [\![1, p]\!]$, let d_i be the number of positive modes in X_i^+ : if
$$X_i^+ = x_{i_1}[n_1] x_{i_2}[n_2] \ldots x_{i_k}[n_k],$$
with $n_j > 0$, then $d_i = k$.
Firstly, let us prove that for all $i \in [\![1, p]\!]$, if $d_i = 0$ then $p_i = 0$. By assumption, $\rho(X) = 0$. In particular X acts trivially on v_α :
$$\begin{aligned}\sum_{i=1}^{p} Q_i P_i X_i v_\alpha &= \sum_{i, d_i = 0} Q_i P_i X_i v_\alpha,\\ &= \sum_{i, d_i=0} p_i Q_i X_i v_\alpha,\\ &= \sum_{i, d_i=0} p_i X_i v_{\mu_i},\\ &= 0.\end{aligned}$$
By assumption, the couples (X_i, μ_i) are pairwise distinct. Thus, because of the structure of the direct sum, for all monomial $\mu \in P$,
$$\sum_{\substack{d_i=0\\ \mu_i=\mu}} p_i X_i v_\mu = 0.$$
Moreover, because of the linear independance of the PBW basis, we get $p_i = 0$ for all i such that $d_i = 0$.
Now, we assume there exists $k > 0$ such that for all i such that $d_i < k$, $p_i = 0$. Let us prove that for all i such that $d_i = k$, we have $p_i = 0$.
For all opposite monomials $A^+ = x_{i_1}[n_1] x_{i_2}[n_2] \ldots x_{i_k}[n_k]$ and $A^- = x_{i_1}[-n_1] x_{i_2}[-n_2] \ldots x_{i_k}[-n_k]$ such that $0 < n_1 \leq \ldots \leq n_k$, for all monomial $B^+ = x_{j_1}[m_1] \ldots x_{j_{k'}}[m_{k'}]$ with $m_i > 0$ and $k' \geq k$, for all $\mu \in P$, a straightforward computation gives :
$$B^+ A^- v_\mu = c_A v_\mu \neq 0 \Longleftrightarrow B^+ = A^+,$$
with c_A a non-zero element in K.
By the induction assumption,
$$X = \sum_{d_i \geq k} Q_i P_i X_i.$$
Let i_0 such that $d_{i_0} = k$. Let A^- be the opposite monomial of $X_{i_0}^+$ (by taking the symmetry :$x_i[n] \mapsto x_i[-n]$). Let $c_i \in K^*$ such that $X_i^+ A^- v_{\mu_i} = c_i v_{\mu_i}$. We get :
$$\begin{aligned}X \cdot (A^- v_0) &= \sum_{d_i \geq k} Q_i P_i X_i \cdot (A^- v_\alpha),\\ &= \sum_{X_i^+ = X_{i_0}^+} Q_i P_i X_i \cdot (A^- v_\alpha),\end{aligned}$$

$$= \sum_{X_i^+ = X_{i_0}^+} p_i Q_i X_i \cdot (A^- v_\alpha),$$
$$= \sum_{X_i^+ = X_{i_0}^+} c_i p_i X_i^- v_{\mu_i}.$$

Again, we get for all $\mu \in P$,

$$\sum_{\substack{X_i^+ = X_{i_0}^+ \\ \mu_i = \mu}} c_i p_i X_i^- v_\mu = 0,$$

and by the linear independence of the PBW basis, we get $c_i p_i = 0$ then $p_i = 0$ for all i such that $X_i^+ = X_{i_0}^+$. This is true for all i_0 such that $d_{i_0} = k$. Hence, $p_i = 0$ for all i such that $d_i = k$.
Hence, by induction, the nullity of p_i for all $i \in [\![1, p]\!]$.
Thus, for all $\alpha \in P$, $P_i \cdot v_\alpha = 0$. Let $p \in \mathbb{N}$ and for all $1 \leq n \leq p$, let $\xi_n = (\xi_{n,1}, \dots, \xi_{n,\ell}) \in \mathbb{C}[\beta]^\ell$,

$$Q = \sum_{n=1}^{p} C_n \prod_{i=1}^{\ell} q^{\xi_{n,i} a_i[0]} \in K[q^{\xi a_j[0]}]_{\xi \in \mathbb{C}[\beta], j \in I}.$$

It is clear that for all $\alpha \in P$,

$$Q \cdot v_\alpha = Q(q^{\xi_1(\alpha, \alpha_1[0])}, \dots, q^{\xi_\ell(\alpha, \alpha_\ell[0])}).$$

Let

$$\alpha := \sum_{i=1}^{\ell} m_i \omega_i,$$

where ω_i is the i-th fundamental weight. We get

$$Q \cdot v_\alpha = \sum_{n=1}^{p} C_n \prod_{i=1}^{\ell} q^{\xi_{n,i} m_i r_i} = 0, \qquad \forall m_1, \dots, m_\ell \in \mathbb{N}.$$

For all $(n, i) \in [\![1, p]\!] \times [\![1, \ell]\!]$, let $X_{n,i} = q^{\xi_{n,i} r_i}$. The variables $X_{n,i}$ and $X_{n',i'}$ are equal if and only if $\xi_{n,i} r_i = \xi_{n',i'} r_{i'}$. However, the ξ_n are pairwise distinct and the r_i are non-zero. Thus, the $X_n = (X_{n,1}, \dots, X_{n,\ell})$ are pairwise distinct: $X_n \neq X_{n'}$ if $n \neq n'$.

$$\sum_{n=1}^{p} C_n \prod_{i=1}^{\ell} X_{n,i}^{m_i} = 0, \qquad \forall m_1, \dots, m_\ell \in \mathbb{N}.$$

For $\ell = 1$, we get a non-zero Vandermonde determinant, this yields to $C_n = 0$ for all $1 \leq n \leq p$.
Suppose the property holds for $\ell - 1$ variables. Let us group the terms in our sum according to the distinct values of the first component $X_{n,1}$. Let $U_1, \dots, U_S$ be the strictly distinct values present in the set $\{X_{1,1}, \dots, X_{p,1}\}$. We can rewrite the sum as:

$$\sum_{s=1}^{S} U_s^{m_1} \left(\sum_{n | X_{n,1} = U_s} C_n X_{n,2}^{m_2} \dots X_{n,\ell}^{m_\ell} \right) = 0$$

Let us fix an arbitrary choice of $(m_2, \dots, m_\ell)$. The equation above holds for all $m_1 \in \mathbb{N}$. Using the exact same Vandermonde argument as in the base case, the linear independence of the powers $U_s^{m_1}$ implies that the term inside the bracket must be zero for each s:

$$\sum_{n | X_{n,1} = U_s} c_n X_{n,2}^{m_2} \dots X_{n,\ell}^{m_\ell} = 0$$

This new equation holds for all $(m_2, \dots, m_\ell) \in \mathbb{N}^{\ell-1}$. Since the original tuples X_n were distinct, the truncated tuples $(X_{n,2}, \dots, X_{n,\ell})$ within the restricted sum (where $X_{n,1}$ is fixed to U_s) are necessarily distinct. By our induction hypothesis on $\ell - 1$ variables, all coefficients C_n within this sub-sum must be zero.

Repeating this for all $s \in [\![1, S]\!]$, we conclude that $C_n = 0$ for all $n \in [\![1, p]\!]$. This implies that P_i is identically zero as an element of the algebra.

This completes the induction step of our main proof. By induction, $P_i = 0$ for all $i \in [\![1, p]\!]$. Hence $X = 0$, which proves that $Ker\rho = \{0\}$ and the representation ρ is faithful. □

Hence, we will denote the $x_i[m]$ instead of $\rho(x_i[m])$ as elements of $End(\bigoplus_{\mu \in P} \pi_\mu)$. The previous proposition gives :

$$\mathcal{H}_{q,t}(\mathfrak{g}) \hookrightarrow End(\bigoplus_{\mu \in P} \pi_\mu).$$

In particular, the $y_i[m]_{i \in I, m<0}$ are algebraically independent as elements of $End(\bigoplus_{\mu \in P} \pi_\mu)$, and so for $(a_i[n])_{i \in I, n \neq 0}$ or $(x_i[n])_{i \in I, n \neq 0}$.

2.3. **Topology on $\mathcal{H}_{q,t}(\mathfrak{g})$.** To define formal series with coefficients in $\mathcal{H}_{q,t}(\mathfrak{g})$ and products of formal series, we have to take a completion of $\mathcal{H}_{q,t}(\mathfrak{g})$. In this subsection, we define a topology on the double deformed Heisenberg algebra and we construct the completion of $\mathcal{H}_{q,t}(\mathfrak{g})$ with respect to this topology. For all $k \geq 0$, we define the ideals

$$I_k := \left\langle \left\{ x_{i_1}[n_1] x_{i_2}[n_2] \dots x_{i_m}[n_m] \ ; \ n_1 \leq n_2 \leq \dots \leq n_m, \ \sum_{j=1}^{m} \max(0, n_j) \geq k \right\} \right\rangle$$

$(I_k)_{k \in \mathbb{Z}}$ be the neighborhood base at 0. This endows a topology on $\mathcal{H}_{q,t}(\mathfrak{g})$.
Let $\widehat{\mathcal{H}}_{q,t}(\mathfrak{g})$ be its completion with respect to this topology.

$$\widehat{\mathcal{H}}_{q,t}(\mathfrak{g}) := \varprojlim_{k \to \infty} (\mathcal{H}_{q,t}(\mathfrak{g}) / I_k)$$

This means that an element $X \in \widehat{\mathcal{H}}_{q,t}(\mathfrak{g})$ can be identified with a coherent sequence $(x_k)_{k \geq 1}$, where $x_k \in \mathcal{H}_{q,t}(\mathfrak{g})/I_k$, such that for all k, the natural projection onto $\mathcal{H}_{q,t}(\mathfrak{g})/I_k$ maps x_{k+1} to x_k.
Concretely, the completion consists of potentially infinite sums of monomials that converge to 0 in the topology. An element $X \in \widehat{\mathcal{H}}_{q,t}(\mathfrak{g})$ is a finite sum or a formal series:

$$X = \sum_{j=0}^{\infty} c_j M_j$$

where $c_j \in K\backslash\{0\}$ and each M_j is a monomial in the generators, satisfying the following convergence condition:
For any integer $N > 0$, all but a finite number of terms in the sum belong to the ideal I_N.
Based on the definition of I_k, this implies that the "annihilation degree" of the terms must tend to infinity:

$$\lim_{j\to\infty}\left(\sum_{p=1}^{d_j}\max(0,n_{j,p})\right)=+\infty,$$

where

$$M_j = x_{i_{j_1}}[n_{j_1}]x_{i_{j_2}}[n_{j_2}]\dots x_{i_{d_j}}[n_{j_{d_j}}].$$

For all $v \in \bigoplus_{\mu\in P}\pi_\mu$, there exists $N \geq 0$ such that for all $k \geq N$,

$$I_k \cdot v = 0$$

Hence, if $X = \sum_{j=0}^{\infty} c_j M_j$, then for all $v \in \bigoplus_{\mu\in P}\pi_\mu$, there exists $M > 0$ such that

$$X\cdot v = \sum_{j=0}^{\infty} c_j M_j \cdot v = \sum_{j=0}^{M} c_j M_j \cdot v \in \bigoplus_{\mu\in P}\pi_\mu$$

We shall now prove that the representation ρ extends to a representation $\widehat{\rho}$ of the completion $\widehat{\mathcal{H}}_{q,t}(\mathfrak{g})$ on $\bigoplus_{\mu\in P}\pi_\mu$ and that this extension is injective. Again, this may be well-known to experts, but we write a proof for completeness.

Proposition 2. *The extension of* $\widehat{\rho}:\widehat{\mathcal{H}}_{q,t}(\mathfrak{g})\to End(\bigoplus_{\mu\in P}\pi_\mu)$ *is injective :*

$$\widehat{\mathcal{H}}_{q,t}(\mathfrak{g}) \hookrightarrow End(\bigoplus_{\mu\in P}\pi_\mu).$$

Proof. We can deduce the injectivity of $\hat{\rho}$ directly from the injectivity of ρ on the subalgebra $\mathcal{H}_{q,t}(\mathfrak{g})$.
Let $M = x_{i_1}[n_1]x_{i_2}[n_2]\dots x_{i_m}[n_m]$ be a monomial in the generators of $\mathcal{H}_{q,t}(\mathfrak{g})$. We define the *annihilation degree* of M, as the sum of its strictly positive modes:

$$\sum_{p=1}^{m}\max(0,n_p).$$

If M does not contain any strictly positive modes (for instance, if M consists only of creation modes, zero modes, or if $M = 1$), we set it to 0.

Let $X = \sum_{j=0}^{\infty} c_j M_j \in \widehat{\mathcal{H}}_{q,t}(\mathfrak{g})$ be a non-zero element such that $\hat{\rho}(X) = 0$. By the convergence condition of the completion, the total number of positive modes (the annihilation degree) in the monomials M_j tends to infinity. This implies that for any integer $k \geq 0$, there are only finitely many terms in the formal series with an annihilation degree equal to k.
Since $X \neq 0$, there is a minimum annihilation degree present in the sum. Let $k \geq 0$ be this minimum degree. We can uniquely decompose X as:

$$X = X_k + X_{>k}$$

where X_k contains all terms of X with an annihilator degree exactly equal to k, and $X_{>k}$ contains the rest of the series (terms with annihilator degree strictly greater than k).

Crucially, because of the limit condition of the topology, X_k is a finite sum. Therefore, X_k is a well-defined, non-zero element of the subalgebra $\mathcal{H}_{q,t}(\mathfrak{g})$.
From our previous proof on the subalgebra $\mathcal{H}_{q,t}(\mathfrak{g})$, the injectivity of ρ relies on the fact that a finite operator X_k with annihilation degree k acts non-trivially on at least one excited state $w = A^- v_\lambda$, where $A^- = x_{i_1}[n_1]x_{i_2}[n_2]\dots x_{i_m}[n_m]$ is a monomial verifiying $\sum_i n_i = -k$. Thus, $X_k \cdot w \neq 0$.
Now, let us consider the tail $X_{>k}$. By definition, every monomial in $X_{>k}$ has an annihilation degree $\geq k+1$. Therefore, $X_{>k} \in I_{k+1}$. By definition of w, any operator from I_{k+1} will annihilate it:

$$X_{>k} \cdot w = 0.$$

Evaluating the full operator X on the vector w, the infinite series truncates exactly to the finite part:

$$X \cdot w = (X_k + X_{>k}) \cdot w = X_k \cdot w + 0 \neq 0$$

This contradicts the assumption that $\hat{\rho}(X) = 0$ on the entire space. We conclude that $Ker\hat{\rho} = \{0\}$, and the extended representation remains strictly injective. □

To simplify the notations, we will denote the completion $\mathcal{H}_{q,t}(\mathfrak{g})$ instead of $\widehat{\mathcal{H}}_{q,t}(\mathfrak{g})$.
Monomials in $\mathcal{H}_{q,t}(\mathfrak{g})$. Let $\mathcal{M}$ be the multiplicative monoid of monomials in the variables $x_i[n]$, $q^{\xi y_i[0]}$, and $e^{\gamma Q_i}$, with $i \in I$, $n \in \mathbb{Z}\backslash\{0\}$, $\xi \in \mathbb{C}[\beta]\backslash\{0\}$, and $\gamma \in \beta^{-1}\mathbb{C}[\beta]$ with coefficients in K. Thus, an element $M \in \mathcal{M}$ is of the form

$$M = \lambda x_{i_1}[n_1]x_{i_2}[n_2]\dots x_{i_m}[n_m]q^{\xi_1 y_1[0]}\dots q^{\xi_\ell y_\ell[0]}e^{\gamma_1 Q_1}\dots e^{\gamma_\ell Q_\ell}$$

with $\lambda \in K^*$, $i_1,\dots,i_m \in I$, $n_1,\dots,n_m \in \mathbb{Z}\backslash\{0\}$, and $\xi_1,\dots,\xi_\ell \in \mathbb{C}[\beta]$, $\gamma_1,\dots,\gamma_\ell \in \beta^{-1}\mathbb{C}[\beta]$.

3. $\mathbf{H}_{q,t}(\mathfrak{g})$ and $\mathbf{W}_{q,t}(\mathfrak{g})$

In this section, we recall the definition of the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$. It was introduced by Frenkel and Reshetikhin in [FR98]. We first introduce some formal power series in $\mathcal{H}_{q,t}(\mathfrak{g})$ and then we recall the definition of the screening operators all due to Frenkel and Reshetikhin. Finally, we recall the definition of the deformed W-algebra as the subalgebra of $\mathcal{H}_{q,t}(\mathfrak{g})$ commuting with the screening operators.

3.1. Some fields in $\mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm1}]]$.

In this section, we introduce some formal power series in $\mathcal{H}_{q,t}(\mathfrak{g})[\![z^{\pm1}]\!]$ due to Frenkel and Reshetikhin ([FR98]), that will be useful in the definition of the screening operators and the deformed W-algebra. We begin by recalling the notion of *fields*.

Definition 2. A *field* $\Phi(z) \in \mathcal{H}_{q,t}(\mathfrak{g})[\![z^{\pm1}]\!]$ is a formal power series

$$\Phi(z) = \sum_{n\in\mathbb{Z}} A_n z^n,$$

such that for all $x \in \bigoplus_{\mu\in P} \pi_\mu$, there exists $N \in \mathbb{Z}$ such that for all $n \leq N$, $A_n x = 0$.

For each $i \in \{1,\dots,\ell\}$, we introduce the following symbols:

$$A_i(z) = t^{2(\rho^\vee,\alpha_i)} q^{-2r(\rho,\alpha_i)+2a_i[0]} \exp\left(\sum_{m\neq0} a_i[m]z^{-m}\right),$$

$$Y_i(z) = t^{2(\rho^\vee,\omega_i)} q^{-2r(\rho,\omega_i)+2y_i[0]} \exp\left(\sum_{m\neq 0} y_i[m]z^{-m}\right).$$

Note that $(\rho^\vee, \alpha_i) = 1, r(\rho, \alpha_i) = r_i$.
$: A_i(z) :$ and $: Y_i(z) :$ are formal series with coefficients in $\mathcal{H}_{q,t}(\mathfrak{g})$. We have :

$$: A_i(z) := t^{2(\rho^\vee,\alpha_i)} q^{-2r(\rho,\alpha_i)+2a_i[0]} : \exp\left(\sum_{m\neq 0} a_i[m]z^{-m}\right) :\in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]],$$

$$: Y_i(z) := t^{2(\rho^\vee,\omega_i)} q^{-2r(\rho,\omega_i)+2y_i[0]} : \exp\left(\sum_{m\neq 0} y_i[m]z^{-m}\right) :\in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]],$$

where $: \cdot :$ stands for the normal ordered product (see [FBZ04]). In general,

$$: \exp\left(\sum_{m\neq 0} x_i[m]z^{-m}\right) : = \exp\left(\sum_{m<0} x_i[m]z^{-m}\right) \exp\left(\sum_{m>0} x_i[m]z^{-m}\right).$$

The *Fourier coefficients* of these formal series lie in $\mathcal{H}_{q,t}$.

Remark 3. It is not the same as defining directly $A_i(z)$ and $Y_i(z)$ as the formal series with the normal ordering. Indeed, a priori, we have :

$$: : Y_i(z) :: Y_i(za) : : \neq : Y_i(z)Y_i(za) :$$

3.2. **Screening operators.** In this section we define the screening currents introduced by Frenkel and Reshetikhin in [FR98]. The $(-1)^{\text{th}}$ Fourier coefficient of the screening currents are called the screening operators, and the deformed W-algebra is a set of fields commuting with the screening operators.
We define $\mathcal{H}'_{q,t}(\mathfrak{g}) := \mathbb{C}[[h]]((\beta)) \otimes_K \mathcal{H}_{q,t}(\mathfrak{g})$.

For each $i \in \{1, \ldots, \ell\}$, $m \in \mathbb{Z}\backslash\{0\}$ define the modes $s_i^{\pm}[m]$ for $m \in \mathbb{Z}$ by the formulas

$$s_i^+[m] = \frac{a_i[m]}{q^{mr_i} - q^{-mr_i}} = x_i[m]\frac{q^m - q^{-m}}{m(q^{mr_i} - q^{-mr_i})} \in \mathcal{H}_{q,t}(\mathfrak{g}), \quad m \neq 0, \tag{9}$$

$$s_i^-[m] = \frac{a_i[m]}{t^m - t^{-m}} = x_i[m]\frac{q^m - q^{-m}}{m(t^m - t^{-m})} \in \mathcal{H}'_{q,t}(\mathfrak{g}), \quad m \neq 0. \tag{10}$$

Remark 4. The element $s_i^-[m]$ does not belong to $\mathcal{H}_{q,t}(\mathfrak{g})$ as it involves negative powers in β.

Now define the following symbols :

$$S_i^+(z) = e^{-Q_i/r_i} \exp\left(\sum_{m\neq 0} s_i^+[m]z^{-m}\right), \tag{11}$$

$$S_i^-(z) = e^{Q_i/\beta} \exp\left(-\sum_{m\neq 0} s_i^-[m]z^{-m}\right) \tag{12}$$

The *screening currents* are the following fields with coefficients in $\mathcal{H}_{q,t}(\mathfrak{g})$:

$$: S_i^+(z) : = e^{-Q_i/r_i} : \exp\left(\sum_{m\neq 0} s_i^+[m]z^{-m}\right) := \sum_{m\in\mathbb{Z}} S_{i,m}^+ z^m \in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]], \tag{13}$$

$$: S_i^-(z) : = e^{Q_i/\beta} : \exp\left(-\sum_{m\neq 0} s_i^-[m]z^{-m}\right) := \sum_{m\in\mathbb{Z}} S_{i,m}^- z^m \in \mathcal{H}'_{q,t}(\mathfrak{g})[[z^{\pm 1}]], \tag{14}$$

where for all $n \in \mathbb{Z}$, for all $i \in \{1, \ldots, \ell\}$, $S_{i,m}^+$ (resp. $S_{i,m}^-$) is seen as a linear map from π_0 to $\pi_{-\beta\alpha_i r/r_i}$ (resp. from π_0 to $\pi_{r\alpha_i}$).

Remark 5. In standard references such as [FR98] and [BP98], the screening currents include a zero-mode factor $z^{\pm s_i^{\mp}[0]}$ and are defined as follows :

$$: S_i^+(z) := e^{-Q_i/r_i} z^{-s_i^+[0]} : \exp\left(\sum_{m\neq 0} s_i^+[m]z^{-m}\right) :$$

$$: S_i^-(z) := e^{Q_i/\beta} z^{s_i^-[0]} : \exp\left(-\sum_{m\neq 0} s_i^-[m]z^{-m}\right) :$$

However, when acting on the module π_0, this operator acts as the identity. Consequently, we omit this factor in our definition. This omission is necessary to ensure mathematical rigor: we require our objects to be formal power series in z with coefficients that are strictly independent of z. Since the status of $z^{\pm s_i^{\mp}[0]}$ as such an object is ambiguous, removing it ensures that the screening currents are well-defined formal series.

They satisfy the difference equations:

$$: S_i^+(zq^{-r_i}) := t^{-2}q^{2r_i} : A_i(z)S_i^+(zq^{r_i}) :, \tag{15}$$

and

$$: S_i^-(zt) := t^{-2}q^{2r_i} : A_i(z)S_i^-(zt^{-1}) : . \tag{16}$$

3.3. **The algebra $\mathbf{H}_{q,t}(\mathfrak{g})$.** In this section, we introduce what Frenkel and Reshetikhin call the *deformed chiral algebra* ([FR98, FR97]) $\mathbf{H}_{q,t}(\mathfrak{g})$. It is a vector space spanned by the monomials on the $Y_i(za)$ and their derivatives. The double deformed W-algebra will be defined as a subspace of $\mathbf{H}_{q,t}(\mathfrak{g})$. However, in this paper we will only consider the monomials on the $Y_i(za)$ without any derivative. It makes sense as this is a subspace of Frenkel and Reshetikhin deformed chiral algebra, but the deformed W- algebra we obtain is not trivial. The second difference with the definition in [FR98] is that in Frenkel and Reshetikhin definition, the spectral parameters lie in $q^{\mathbb{Z}}t^{\mathbb{Z}}$. In our context, we allow all spectral parameters in $\mathbb{C}^*q^{\mathbb{Z}}t^{\mathbb{Z}}$ so that the limit $t \to 1$ contain more q-characters (this will be studied in an upcoming paper).

Let $\mathbf{H}_{q,t}(\mathfrak{g}) \subset \mathcal{H}_{q,t}[\![z^{\pm 1}]\!]$ be the vector space spanned by formal power series of the form

$$: Y_{i_1}(za_1)^{\epsilon_1} \dots Y_{i_m}(za_m)^{\epsilon_m} : \in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]]$$

for $m \geq 1$, $\epsilon_i = \pm 1$, $a_1, \dots, a_m \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$.

Lemma 2. The normal ordering $: Y_{i_1}(za_1)^{\epsilon_1} \dots Y_{i_m}(za_m)^{\epsilon_m} :$ is independant on the ordering of the factors.

Remark 6. The original definition introduced by Frenkel and Reshetikhin in [FR98] is the K-vector space spanned by the monomials of the form :

$$: \partial_z^{n_1} Y_{i_1}(zq^{j_1}t^{k_1})^{\epsilon_1} \dots \partial_z^{n_m} Y_{i_m}(zq^{j_m}t^{k_m})^{\epsilon_m} : \in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]]$$

with $\epsilon_i = \pm 1$.

Definition 3. Let $\mathbf{M}$ be the following set :

$$\mathbf{M} := \{: Y_{i_1}(za_1)^{\epsilon_1} \dots Y_{i_m}(za_m)^{\epsilon_m} :\} \subset \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]]$$

for $m \geq 1$, $\epsilon_i = \pm 1$, $a_1, \dots, a_m \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$.
An element $m \in \mathbf{M}$ is called a *monomial* in the $Y_i(za)^{\pm 1}$.

Proposition 3. *The map*

$$\left\{ \begin{array}{ccc} K[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}} & \longrightarrow & \mathcal{H}_{q,t}[\![z^{\pm 1}]\!] \\ \sum_{p=1}^{d} \lambda_p \prod_{i=1}^{d_p} Y_{j_{p,i}, a_{p,i}} & \longmapsto & \sum_{p=1}^{d} \lambda_p : \prod_{i=1}^{d_p} Y_{j_{p,i}}(za_{p,i}) : \end{array} \right. \tag{17}$$

is an injective linear map. In particular, the fields $Y_i(za)_{i \in I, a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$ are algebraically independent with respect to the normally ordered product.

Proof. Let $(M_p)_p$ be a finite set of distinct monomials such that

$$\sum_{p=1}^{N} \lambda_p M_p = 0$$

We will prove that for all p, $\lambda_p = 0$. Let

$$M_p =: Y_{j_{p,1}}(za_{p,1})^{m_{p,1}} Y_{j_{p,2}}(za_{p,2})^{m_{p,2}} \dots Y_{j_{p,d_p}}(za_{p,d_p})^{m_{p,d_p}} :$$

We define the following left-ideal of the algebra $\mathcal{H}_{q,t}(\mathfrak{g})$:

$$\mathcal{H}_{q,t}^{+}(\mathfrak{g}) := \langle x_i[n], (q^{\xi y_i[0]} - 1), (t^{y_i[0]} - 1) \rangle_{i \in I, \xi \in \mathbb{C}, n > 0}$$

We define the canonical projection $\Pi : \mathcal{H}_{q,t}(\mathfrak{g}) \longrightarrow \mathcal{H}_{q,t}(\mathfrak{g}) / \mathcal{H}_{q,t}^{+}(\mathfrak{g})$. It is a homomorphism of left $\mathcal{H}_{q,t}(\mathfrak{g})-$*module*. We define its extension to formal series with coefficients in $\mathcal{H}_{q,t}(\mathfrak{g})$ as follows :

$$\Pi : \left\{ \begin{array}{ccc} \mathbf{H}_{q,t}(\mathfrak{g}) & \longrightarrow & \mathbf{H}_{q,t}(\mathfrak{g}) \\ \sum_{n \in \mathbb{Z}} A_n z^n & \longmapsto & \sum_{n \in \mathbb{Z}} \Pi(A_n) z^n \end{array} \right.$$

Then

$$\sum_{p} \lambda_p \Pi(M_p) = 0.$$

Furthermore, for all $i \in I$, for all $a \in q^{\mathbb{Z}}t^{\mathbb{Z}}$,

$$\Pi(Y_i(za)) = \Pi\left(t^{2(\rho^\vee,\omega_i)}q^{-2r(\rho,\omega_i)+2y_i[0]}\exp\left(\sum_{n>0} y_i[-n](az)^n\right)\exp\left(\sum_{n>0} y_i[n](az)^{-n}\right)\right)$$
$$= C\exp\left(\sum_{n>0} y_i[-n](az)^n\right),$$

with $C = t^{2(\rho^\vee,\omega_i)}q^{-2r(\rho,\omega_i)} \in K^*$.
Thus,

$$\Pi(M_p) = C_p \exp\left(\sum_{n>0}\sum_{r=1}^{d_p} m_{p,r}y_{j_{p,r}}[-n]z^n {a_{p,r}}^n\right)$$
$$= C_p\left(1 + z\sum_{r=1}^{d_p} m_{p,r}y_{j_{p,r}}[-1]a_{p,r} + z^2\left[\sum_{r=1}^{d_p} m_{p,r}y_{j_{p,r}}[-2]{a_{p,r}}^2 + \frac{1}{2}\left(\sum_{r=1}^{d_p} m_{p,r}y_{j_{p,r}}[-1]a_{p,r}\right)^2\right] + \dots\right)$$
$$= C_p\sum_{n>0} z^n \sum_{\substack{1\le k\le n\\ n_1+\dots+n_k=n\\ n_i>0}} \frac{1}{k!}\prod_{j=1}^{k}\left(\sum_{r=1}^{d_p} m_{p,r}y_{j_{p,r}}[-n_j]{a_{p,r}}^{n_j}\right),$$

with $C_p \in K^*$ a non-zero constant depending of q, t.
It is a linear combination of elements of the form

$$z^n y_{i_1}[-n_1]^{k_1} y_{i_2}[-n_2]^{k_2}\dots y_{i_s}[-n_s]^{k_s},$$

but the $(y_i[-m])_{i\in I, m>0}$ are algebraically independent in $\mathcal{H}_{q,t}(\mathfrak{g})$. Then for all finite set of tuples $(i_u, n_u, k_u) \in I \times \mathbb{N}^* \times \mathbb{N}^*$ such that the $(i_u, n_u) \in I \times \mathbb{N}^*$ are pairwise distinct, $1 \le u \le s$,

$$\sum_p \lambda_p C_p \frac{1}{(\sum_u k_u)!}\prod_{u=1}^{s}\left(\sum_{r|j_{p,r}=i_u} m_{p,r}{a_{p,r}}^{n_u}\right)^{k_u} = 0.$$

Then for all finite set of couples $(i_u, n_u, k_u) \in I \times \mathbb{N}^* \times \mathbb{N}^*$ such that the $(i_u, n_u) \in I \times \mathbb{N}^*$ are pairwise distinct, $1 \le u \le s$,

$$\sum_p \lambda_p C_p \prod_{u=1}^{s}\left(\sum_{r|j_{p,r}=i_u} m_{p,r}{a_{p,r}}^{n_u}\right)^{k_u} = 0.$$

Let $S_{p,u}=\sum_{r|j_{p,r}=i_u} m_{p,r}{a_{p,r}}^{n_u}$. We have

$$\sum_{b_1,\ldots,b_s\in K^*} b_1^{k_1}b_2^{k_2}\ldots b_s^{k_s}\sum_{\substack{p\\ \forall u, S_{p,u}=b_u}}\lambda_pC_p=0.$$

This equality holds for all $(k_u)\in(\mathbb{N}^*)^s$. It is a Vandermonde system and it implies

$$\sum_{\substack{p\\ \forall u, S_{p,u}=b_u}}\lambda_pC_p=0$$

for all $(b_1,\ldots,b_s)\in(K^*)^s$.
Let $p\in[[1,N]]$. We want to conclude that $\lambda_p=0$. So we want to prove that this sum contains at most one term. We assume that there exists p' such that for all set of pairwise distinct couples $(i_u,n_u)_u\in(I\times\mathbb{N}^*)^s$, for all $u\in[[1,s]]$, $S_{p,u}=S_{p',u}$. Then for all set of pairwise distinct couples $(i_u,n_u)_u\in(I\times\mathbb{N}^*)^s$, for all $u\in[[1,s]]$,

$$\sum_{r|j_{p,r}=i_u} m_{p,r}{a_{p,r}}^{n_u}=\sum_{r|j_{p',r}=i_u} m_{p',r}{a_{p',r}}^{n_u}.$$

Then

$$\sum_{a\in K^*}\sum_{\substack{r|j_{p,r}=i_u\\ a_{p,r}=a}} m_{p,r}a^{n_u}=\sum_{a\in K^*}\sum_{\substack{r|j_{p',r}=i_u\\ a_{p',r}=a}} m_{p',r}a^{n_u},$$

and we obtain the following Vandermonde system

$$\sum_{a\in K^*}\left(\sum_{\substack{r|j_{p,r}=i_u\\ a_{p,r}=a}} m_{p,r}-\sum_{\substack{r|j_{p',r}=i_u\\ a_{p',r}=a}} m_{p',r}\right)a^{n_u}=0.$$

We fix i_u and we vary n_u to obtain for all $u\in[[1,s]]$, for all i_u,

$$\sum_{\substack{r|j_{p,r}=i_u\\ a_{p,r}=a}} m_{p,r}=\sum_{\substack{r|j_{p',r}=i_u\\ a_{p',r}=a}} m_{p',r}.$$

Each sum contains at most one term. It implies $M_p=M_{p'}$ and $p=p'$.
Finally, we can fix a $s-$tuple $(i_u,n_u,b_u)_u\in(I\times\mathbb{N}^*\times K^*)^s$ such that λ_p is the unique term in the sum $\sum_{\substack{p\\ \forall u, S_{p,u}=b_u}}\lambda_pC_p=0$. Hence $\lambda_pC_p=0$, and

$$\lambda_p=0.$$

□

Corollary 1. *(1) The map (17) is an isomorphism of vector spaces :*

$$\mathbf{H}_{q,t}(\mathfrak{g})\simeq K[Y_{i,a}^{\pm1}]_{i\in I,a\in\mathbb{C}^*q^{\mathbb{Z}}t^{\mathbb{Z}}},$$

sending $Y_{i,a}^{\pm1}\mapsto Y_i(za)^{\pm1}$.

(2) *The* $(A_i(za))_{i\in I, a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$ *are algebraically independent in* $\mathbf{H}_{q,t}(\mathfrak{g})$ *with respect to the normally ordered product.*

Proof. (1) The map (17) is surjective by definition of $\mathbf{H}_{q,t}(\mathfrak{g})$.

(2) The proof is the same, replacing $Y_i(za)$ by $A_i(za)$, because of the algebraic independance of the $(a_i[n])_{i\in I, n\in\mathbb{Z}}$ in $\mathcal{H}_{q,t}(\mathfrak{g})$. □

Definition 4. We also define a degree. We denote $d_{i,a}$ the degree of a field $\Phi(z)$ by taking its $Y_{i,a}$-degree in the image of $\Phi(z)$ in $K[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$.

3.4. **The deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$.** We shall now define the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ as introduced by Frenkel and Reshetikhin in [FR98]. Firstly, we define the *screening operators* as follows :

$$S_i^+ := S_{i,-1}^+ \in \mathcal{H}_{q,t}(\mathfrak{g}), \qquad S_i^- := S_{i,-1}^- \in \mathcal{H}'_{q,t}(\mathfrak{g}),$$

where

$$S_i^+(w) = \sum_{m\in\mathbb{Z}} S_{i,m}^+ w^{-m} \in \mathcal{H}_{q,t}(\mathfrak{g})[[w^{\pm 1}]], \qquad S_i^-(w) = \sum_{m\in\mathbb{Z}} S_{i,m}^- w^{-m} \in \mathcal{H}_{q,t}(\mathfrak{g})'[[w^{\pm 1}]].$$

It is the Fourier coefficient in front of w^{-1}. It is also called *the residue* at 0 of $S_i^{\pm}(w)$. We say that a field $A(z) = \sum A_n z^{-n} \in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]]$ commutes with an operator $B \in \mathcal{H}'_{q,t}(\mathfrak{g})$ if for all $n\in\mathbb{Z}$, A_n commutes with B.

Definition 5. Let $\mathbf{W}_{q,t}(\mathfrak{g})$ be the vector subspace of $\mathbf{H}_{q,t}(\mathfrak{g})$ of fields commuting with the screening operators $S_i^{\pm}$.
The *deformed W-algebra associated to the Lie algebra* $\mathfrak{g}$ is the subalgebra of $\mathcal{H}_{q,t}(\mathfrak{g})$ generated by the Fourier coefficients of the fields in $\mathbf{W}_{q,t}(\mathfrak{g})$ and is denoted $\mathcal{W}_{q,t}(\mathfrak{g})$.
By abuse of notation, both $\mathcal{W}_{q,t}(\mathfrak{g})$ and $\mathbf{W}_{q,t}(\mathfrak{g})$ are called *the deformed W-algebra*.

3.5. **Some Computations.** In this section, we provide the fundamental computations required for the next sections. In particular, we derive explicit expressions for the coefficients within the fields of $\mathbf{W}_{q,t}(\mathfrak{g})$.

3.5.1. *Operator Product Expansion (OPE) and difference relations.* We recall the following OPEs from [FR98, BP98] for all $i,j\in I$, $i\neq j$:

$$:Y_i(z)::S_i^+(w): = t^{-2}\left(\frac{1-t\frac{w}{z}}{1-t^{-1}\frac{w}{z}}\right):Y_i(z)S_i^+(w): \qquad :S_i^+(w)::Y_i(z): = \frac{1-t^{-1}\frac{z}{w}}{1-t\frac{z}{w}}:S_i^+(w)Y_i(z):$$

$$:Y_i(z)::S_i^-(w): = q^{2r_i}\left(\frac{1-q^{-r_i}\frac{w}{z}}{1-q^{r_i}\frac{w}{z}}\right):Y_i(z)S_i^-(w): \qquad :S_i^-(w)::Y_i(z): = \frac{1-q^{r_i}\frac{z}{w}}{1-q^{-r_i}\frac{z}{w}}:S_i^-(w)Y_i(z):$$

$$:Y_i(z)::S_j^{\pm}(w): =: Y_i(z)S_j^{\pm}(w) :=: S_j^{\pm}(w)::Y_i(z): \qquad (i\neq j)$$

The first and second (resp. third and fourth) expressions have to be understood as formal power series in positive powers of $\frac{w}{z}$ (resp. $\frac{z}{w}$). We remark that the first two rational functions are the same as third and fourth.
To construct elements in $\mathbf{W}_{q,t}(\mathfrak{g})$ means constructing elements $\Phi(z)$ such that for all $i\in I$,

$$\mathrm{Res}_w[\Phi(z), S_i^{\pm}(w)] = 0.$$

Thus, we will use the difference relations (15), (16) to get :

$$\delta\left(\frac{w}{zq^{r_i}}\right) : Y_i(z)^{-1}S_i^-(w) := t^{-2}q^{2r_i}\delta\left(\frac{w}{zq^{r_i}}\right) : Y_i(z)^{-1}A_i(zq^{r_i}t^{-1})S_i^-(wt^{-2}) \tag{18}$$

and

$$\delta\left(\frac{w}{zt^{-1}}\right) : Y_i(z)^{-1}S_i^+(w) := t^{-2}q^{2r_i}\delta\left(\frac{w}{zt^{-1}}\right) : Y_i(z)^{-1}A_i(zq^{r_i}t^{-1})S_i^+(wq^{2r_i}) \tag{19}$$

3.5.2. *Commutators with the screening.* We deduce the following commutator: let

$$M(z) =: \prod_{j=1}^{n} Y_{i_j}(za_j)^{\varepsilon_j m_j} :,$$

with $\varepsilon_i = \pm 1$, $m_i > 0$, $a_i \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$. We have for all $j \in I$:

$$[: M(z) :, : S_j^+(w) :] = \left[i_{z,w} \prod_{k|i_k=j} \left(\frac{1 - t^{-\varepsilon_k} \frac{za_k}{w}}{1 - t^{\varepsilon_k} \frac{za_k}{w}} \right)^{m_k} - i_{w,z} \prod_{k|i_k=j} \left(\frac{1 - t^{-\varepsilon_k} \frac{za_k}{w}}{1 - t^{\varepsilon_k} \frac{za_k}{w}} \right)^{m_k} \right] : M(z)S_j^+(w) :,$$

$$[: M(z) :, : S_j^-(w) :] = \left[i_{z,w} \prod_{k|i_k=j} \left(\frac{1 - q^{\varepsilon_k r_j} \frac{za_k}{w}}{1 - q^{-\varepsilon_k r_j} \frac{za_k}{w}} \right)^{m_k} - i_{w,z} \prod_{k|i_k=j} \left(\frac{1 - q^{\varepsilon_k r_j} \frac{za_k}{w}}{1 - q^{-\varepsilon_k r_j} \frac{za_k}{w}} \right)^{m_k} \right] : M(z)S_j^-(w) :,$$

where $i_{z,w}(F)$ (resp $i_{w,z}(F)$) is the formal power series expansion in $|w| < |z|$ (resp. $|w| > |z|$) of the rational function F.
For example, if $i \in I$,

$$[: Y_i(z) :, : S_i^-(w) :] = \delta\left(\frac{w}{z}q^{r_i}\right)(q^{2r_i} - 1) : Y_i(z)S_i^-(w) : .$$

To simplify the computations, we will put some hypotheses on the monomials.

Let : $M(z) :\in \mathcal{H}_{q,t}(\mathfrak{g})[[z^{\pm 1}]]$ such that

$$M(z) = \prod_{i=1}^{r} Y_{i_i}(za_i) \prod_{j=1}^{s} Y_{j_j}(zb_j)^{-1}$$

with $a_i, b_j \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$, $i_i, j_j \in I$, and such that

- for all $1 \leq i \neq j \leq r$, $(i_i, a_i) \neq (i_j, a_j)$.
- for all $1 \leq i \neq j \leq s$, $(j_i, b_i) \neq (j_j, b_j)$.
- for all $1 \leq i \leq r$, for all $1 \leq j \leq s$, $(i_i, a_i) \neq (j_j, b_j)$.
- for all $1 \leq i \leq r$, for all $1 \leq j \leq s$, $(i_i, a_i) \neq (j_j, b_j q^{2r_{i_i}})$.
- for all $1 \leq i \leq r$, for all $1 \leq j \leq s$, $(i_i, a_i) \neq (j_j, b_j t^{-2})$.
- for all $1 \leq i \leq r$, for all $1 \leq j \leq s$, $(i_i, a_i) \neq (j_j, b_j q^{2r_{i_i}} t^{-2})$.
- for all $1 \leq i, j \leq r$, if $(i_i, a_i) = (i_j, a_j q^{2r_{i_i}} t^{-2})$, then there exists $1 \leq u, v \leq r$ such that $(i_u, a_u) = (i_i, a_i q^{-2r_{i_i}})$ and $(i_v, a_v) = (i_i, a_i t^2)$.
- for all $1 \leq i, j \leq s$, if $(j_i, b_i) = (j_j, b_j q^{2r_{j_i}} t^{-2})$, then there exists $1 \leq u, v \leq r$ such that $(j_u, b_u) = (j_i, b_i q^{-2r_{j_i}})$ and $(j_v, b_v) = (j_i, b_i t^2)$.

Remark 7. We will define later that such a monomial is called *generic* and *regular*.
The two first items express the *genericity*, the third one expresses the fact the monomial is reduced (we do not have terms of the form $Y_i(za)Y_i(za)^{-1}$).
All the other items express the *regularity* of the monomial.

Let $R \subset [\![1,r]\!]$ (resp. $S \subset [\![1,s]\!]$) be the set of indices i such that for all $1 \leq j \leq r$, $(i_j, a_j) \neq (i_i, a_i q^{-2r_{i_i}})$ (resp. for all $1 \leq j \leq s$, $(j_j, b_j) \neq (j_i, b_i q^{2r_{j_i}})$). Then we have the following commutators :

$$[: M(z) :, : S_k^-(w) :] = \left[\sum_{i \in R | i_i = k} C_{k,i}^+ \delta\left(\frac{w}{za_i} q^{r_k}\right) + \sum_{j \in S | j_j = k} C_{k,j}^- \delta\left(\frac{w}{zb_j} q^{-r_k}\right) \right] : M(z) S_k^-(w) :$$

The coefficients $C_{k,i}^{\pm}$ are given by the partial fraction decomposition of the rational functions written above.

3.5.3. *Which coefficients for the monomials in the fields in* $\mathbf{W}_{q,t}(\mathfrak{g})$ *?* We treat here the *generic* and *regular* case. As $Y_j(za)$ interferes with the screening operator $S_i^{\pm}$ if and only if $i = j$, we can consider a monomial M_1 expressed only in terms of the $Y_i^{\pm 1}$. Let

$$: M_1 :=: Y_i(za) Y_i(zb_1) \dots Y_i(zb_k) Y_i(zc_1)^{-1} \dots Y_i(zc_\ell)^{-1} :,$$

and

$$: M_2 :=: M_1 A_i(zaq^{-r_i}t)^{-1} :=: Y_i(zaq^{-2r_i}t^2)^{-1} Y_i(zb_1) \dots Y_i(zb_k) Y_i(zc_1)^{-1} \dots Y_i(zc_\ell)^{-1} : .$$

The presence of M_2 in our field serves to cancel the residue of the delta-function $\delta(\frac{w}{z} q^{r_i} a^{-1})$ in the expression of $[: M_1 :, S_i^-]$.
By a straightforward computation, the term in front of $\delta(\frac{w}{z} q^{r_i} a^{-1})$ in $[: M_1 :, S_i^-]$ is

$$q^{2r_i(k-\ell+1)}(1 - q^{-2r_i}) \prod_{j=1}^{k} \frac{1 - q^{-2r_i} ab_j^{-1}}{1 - ab_j^{-1}} \prod_{u=1}^{\ell} \frac{1 - ac_u^{-1}}{1 - q^{-2r_i} ac_u^{-1}},$$

and the term in front of $\delta(\frac{w}{z} q^{r_i} t^{-2} a^{-1})$ in $[: M_2 :, S_i^-]$ is

$$q^{2r_i(k-\ell-1)}(1 - q^{2r_i}) \prod_{j=1}^{k} \frac{1 - q^{-2r_i} t^2 ab_j^{-1}}{1 - t^2 ab_j^{-1}} \prod_{u=1}^{\ell} \frac{1 - t^2 ac_u^{-1}}{1 - q^{-2r_i} t^2 ac_u^{-1}}.$$

To cancel the residue of the delta function $\delta(\frac{w}{z} q^{r_i} a^{-1})$ in $[: M_1 :, S_i^-]$, the difference relation implies that the coefficient λ_{M_2} in front of $: M_2 :$ must therefore satisfy:

$$\lambda_{M_2} \times q^{2r_i} q^{2r_i(k-\ell-1)}(1 - q^{2r_i}) \prod_{j=1}^{k} \frac{1 - q^{-2r_i} t^2 ab_j^{-1}}{1 - t^2 ab_j^{-1}} \prod_{u=1}^{\ell} \frac{1 - t^2 ac_u^{-1}}{1 - q^{-2r_i} t^2 ac_u^{-1}}$$

$$= -\lambda_{M_1} \times q^{2r_i(k-\ell+1)}(1 - q^{-2r_i}) \prod_{j=1}^{k} \frac{1 - q^{-2r_i} ab_j^{-1}}{1 - ab_j^{-1}} \prod_{u=1}^{\ell} \frac{1 - ac_u^{-1}}{1 - q^{-2r_i} ac_u^{-1}}.$$

Thus

$$\boxed{\lambda_{M_2} = \lambda_{M_1} \prod_{j=1}^{k} \frac{(1 - q^{-2r_i} ab_j^{-1})(1 - t^2 ab_j^{-1})}{(1 - ab_j^{-1})(1 - q^{-2r_i} t^2 ab_j^{-1})} \prod_{u=1}^{\ell} \frac{(1 - ac_u^{-1})(1 - q^{-2r_i} t^2 ac_u^{-1})}{(1 - q^{-2r_i} ac_u^{-1})(1 - t^2 ac_u^{-1})}.} \tag{20}$$

Remark 8. We recognize a product of $\mathcal{S}$-functions defined by Kimura and Pestun in [KP18a] (see equation (3.45) in section 3.5.2).

Proposition 4. *Assume $\lambda_{M_1} \in K^\times$. If the parameters $a, b_j, c_u \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$ avoid the exact resonance set $ab_j^{-1}, ac_u^{-1} \notin \{1, q^{2r_i}, t^{-2}, q^{2r_i}t^{-2}\}$ for all j and u, then $\lambda_{M_2} \in K^\times$. Moreover, if $ab_j^{-1}, ac_u^{-1} \notin \{t^{-1}, q^{2r_i}t^{-1}\}$, then $\lambda_{M_2}/\lambda_{M_1} \in \mathbb{Q}_{>0} + \beta K$.*

Proof. It is enough to show that each block in the product evaluates to a non-zero complex constant as $(h, \beta) \to (0, 0)$. Let x denote either ab_j^{-1} or ac_u^{-1}. By definition, $x = Ce^{h(m+n\beta)}$ for some $C \in \mathbb{C}^*$ and $m, n \in \mathbb{Z}$.
If $C \neq 1$, all factors in the block are of the form $1 - C + O(h)$. Since $1 - C \neq 0$, the block is invertible in K.
Now assume $C = 1$. Consider the block associated with b_j:

$$N_b(x) = \frac{(1 - e^{-2r_i h}x)(1 - e^{2\beta h}x)}{(1 - x)(1 - e^{(-2r_i + 2\beta)h}x)}$$

Using $1 - e^{hA} = -hA(1 + O(h))$, we factor out h^2 from both the numerator and the denominator. The h^2 terms cancel perfectly, and the leading order as $h \to 0$ leaves a rational function in β:

$$D_b(\beta) = \frac{(m + n\beta - 2r_i)(m + (n+2)\beta)}{(m + n\beta)(m - 2r_i + (n+2)\beta)}$$

Evaluating at $\beta = 0$ gives $D_b(0) = 1 \neq 0$, unless $m = 0$ or $m = 2r_i$.
If $m = 0$, a factor of β cancels out, yielding $\lim_{\beta \to 0} D_b(\beta) = \frac{n+2}{n}$. The hypothesis in the Proposition ensures $n \notin \{0, -2\}$, making this limit a well-defined non-zero constant strictly positive if $n \neq -1$.
If $m = 2r_i$, a similar cancellation gives $\lim_{\beta \to 0} D_b(\beta) = \frac{n}{n+2}$, which is again finite, non-zero since $n \notin \{0, -2\}$ by assumption and strictly positive if $n \neq -1$.
The block associated with c_u is simply the reciprocal, $N_c(y) = N_b(y)^{-1}$. By the same analysis, its limit is also a non-zero constant. Since the h^2 singularity is fully removed and the evaluation at $(h, \beta) = (0, 0)$ is strictly non-zero, every block is invertible in K. Thus, $\lambda_{M_2} \in K^\times$. ☐

Example 1. *In the case of $\mathfrak{g} = \mathfrak{sl}_2$, we have $r_1 = 1$. Let us consider the field*

$$T(z) =: Y(z)Y(zq^{-2}) : +\lambda_2 : Y(z)Y(zq^{-4}t^2)^{-1} : +\lambda_3 : Y_1(zq^{-2}t^2)^{-1}Y(zq^{-4}t^2)^{-1}.$$

We have $M_1 =: Y(z)Y(zq^{-2}) :$ and $M_2 =: Y(z)Y(zq^{-4}t^2)^{-1} :$. By (20), we have

$$\lambda_2 = \lambda_1 \frac{(q + q^{-1})(qt^{-1} - q^{-1}t)}{q^2t^{-1} - q^{-2}t},$$

then we get

$$\lambda_3 = \lambda_2 \frac{q^2t^{-1} - q^{-2}t}{(q + q^{-1})(qt^{-1} - q^{-1}t)} = \lambda_1.$$

Finally, setting $\lambda_1 = 1$, we get the field

$$T(z) =: Y(z)Y(zq^{-2}) : + \frac{(q+q^{-1})(qt^{-1}-q^{-1}t)}{q^2t^{-1}-q^{-2}t} : Y(z)Y(zq^{-4}t^2)^{-1} :$$

$$+ : Y_1(zq^{-2}t^2)^{-1}Y(zq^{-4}t^2)^{-1} :,$$

and we recover the interpolating function introduced in [FR96, FR98] *and constructed in a more elementary way in* [FH11] *(Section 4.3).*

4. Construction of fields in $\mathbf{W}_{q,t}(\mathfrak{g})$

In this section, we explicitly construct elements in $\mathbf{W}_{q,t}(\mathfrak{g})$. We introduce an algorithm inspired by the Frenkel-Mukhin algorithm [FM01]. First, we describe this algorithm which, given a *generic dominant regular* monomial $m \in \mathbf{M}$ (defined below), produces a field in $\mathbf{H}_{q,t}(\mathfrak{g})$. Then, we prove that this algorithm is well-defined and that the resulting field lies in $\mathbf{W}_{q,t}(\mathfrak{g})$. Finally, we give examples of the results of this algorithm for various types of Lie algebras.

The Frenkel-Mukhin algorithm is inspired by classical Lie algebra representation theory, where characters are constructed using the Weyl group action by subtracting simple roots from the weights.
The two main differences between the Frenkel-Mukhin algorithm and the one we propose are the following:

- First, at each step, we do not define an i-expansion as Frenkel and Mukhin do in [FM01]. Instead, we expand each *admissible* variable $Y_i(za)$ (defined below) one by one.
- The second and main difference is the definition of the coefficients associated with the new monomials created at each step of the algorithm. Indeed, in [FM01], the coefficients are defined as maxima, while here we define them explicitly as a quotient of residues of rational functions using formula (21).

From now on, to simplify notations, we will drop the normal ordering symbol : :. For all monomials $m, m' \in \mathbf{M}$, the notation mm' will always denote the normal ordered product $: mm' :$.

4.1. Description of the algorithm.

Definition 6. We say that a monomial $M \in \mathbf{M}$ is *dominant* if for all $i \in I$, $a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$, $d_{i,a}(M) \geq 0$.

Remark 9. It is well-defined since the variables $Y_i(za)$ are algebraically independent in $\mathcal{H}_{q,t}(\mathfrak{g})$ with respect to the normal ordered product.

Let us firstly introduce some condition we will impose on our fields in this framework.

Definition 7. We say that a monomial $M \in \mathbf{M}$ is *generic* if for all $i \in I, a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$, $d_{i,a}(M) \in \{-1, 0, 1\}$.
We say that a monomial $m =: \prod_i Y_{j_i}(za_i)^{\varepsilon_i} :$ is *regular* when for all $a \in K$, for all $i \in I$,

- If $d_{i,a}(m) > 0$ then $d_{i,aq^{-2r_i}}(m), d_{i,at^2}(m) \geq 0$
- If $d_{i,a}(m) > 0$ and $d_{i,aq^{-2r_i}t^2}(m) > 0$ (resp $d_{i,a}(m) < 0$ and $d_{i,aq^{-2r_i}t^2}(m) < 0$), then $d_{i,aq^{-2r_i}}(m) > 0$ and $d_{i,at^2}(m) > 0$ (resp. $d_{i,aq^{-2r_i}}(m) < 0$ and $d_{i,at^2}(m) < 0$)

Let m be a regular monomial. We say that $Y_i(za)$ is *admissible* in m if

$$d_{i,a}(m) > 0 \quad \text{and} \quad d_{i,aq^{-2r_i}}(m) = d_{i,at^2}(m) = 0$$

Remark 10. Roughly speaking, to be regular means not to contain $Y_i(za)Y_i(zaq^{-2r_i})^{-1}$ nor $Y_i(za)Y_i(zat^2)^{-1}$, and to contain $Y_i(za)Y_i(zaq^{-2r_i}t^2)$ only if it also contains $Y_i(zaq^{-2r_i})Y_i(zat^2)$.

Now let us describe an algorithm which, given a generic dominant regular monomial m, gives an element $T(m) \in \mathbf{W}_{q,t}(\mathfrak{g})$ such that m is the unique dominant monomial appearing in the expression of $T(m)$:
We want to obtain a list of monomials with their associated coefficients.

- ***Step 0 :*** We get the first monomial to be m with the coefficient 1.
- ***At each step :*** At the previous step, we obtained a list of monomials with their coefficients : $\lambda_1 m_1, \lambda_2 m_2, \dots, \lambda_p m_p$.
  If any m_i is not regular or not generic then the algorithm *fails*, it stops here. Else, for each $1 \leq i \leq p$, for each admissible $Y_j(za)$ in m_i then if it was not already created, we create the monomial $m'_{i,(j,a)} =: m_i A_j(zaq^{-r_j}t)^{-1}$:.
  We shall now compute the coefficient associated to $m'_{i,(j,a)}$. We set :
  $$m_i =: Y_j(za)Y_j(zb_1)\dots Y_j(zb_k)Y_j(zc_1)^{-1}\dots Y_j(zc_r)^{-1}n_i :$$
  and
  $$m'_{i,(j,a)} =: Y_j(zaq^{-2r_j}t^2)^{-1}Y_j(zb_1)\dots Y_j(zb_k)Y_j(zc_1)^{-1}\dots Y_j(zc_r)^{-1}n'_{i,(j,a)} : .$$
  where n_i and $n'_{i,(j,a)}$ are monomials in the variables $Y_u(zd)$ for $u \neq j$ and $d \in K$.
  The coefficient $\lambda_{i,k}$ associated to $m'_{i,(j,a)}$ is defined as in (20) :
  $$\boxed{\lambda_{m'_{i,(j,a)}} = \lambda_{m_i} \prod_{j=1}^{k} \frac{(1-q^{-2r_i}ab_j^{-1})(1-t^2ab_j^{-1})}{(1-ab_j^{-1})(1-q^{-2r_i}t^2ab_j^{-1})} \prod_{u=1}^{\ell} \frac{(1-ac_u^{-1})(1-q^{-2r_i}t^2ac_u^{-1})}{(1-q^{-2r_i}ac_u^{-1})(1-t^2ac_u^{-1})}.} \quad (21)$$

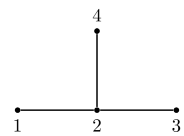
We repeat the steps until no new monomial can be created. We denote by $T(m)$ the sum of all the monomials created with their associated coefficients. We have seen that this coefficient lies in K (see Proposition 4).

Definition 8. Let $m \in \mathbf{M}$ be a dominant generic regular monomial. We define $\mathbf{A}(m)$ to be the set of monomials created by the algorithm (including the initial dominant generic regular monomial m).
For all $M \in \mathbf{A}(m)$, $i \in I$, and $a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$. We say that the transformation $A_i(za)^{-1}$ is *admissible* in M if $Y_i(zaq^{r_i}t^{-1})$ is admissible in M.
Let m be a dominant generic regular monomial. We say that $m' \in \mathbf{M}$ *appears* in the algorithm if
$$m' \in \mathbf{A}(m).$$

Example 2. *For $\mathfrak{g} = \mathfrak{sl}_2$, we can construct a field $T(z) \in \mathbf{W}_{q,t}(\mathfrak{g})$ starting from the dominant monomial $m = Y(z)Y(zq^{-2})Y(zt^2)$:*
$$T(z) =: Y(z)Y(zq^{-2})Y(zt^2) : +\lambda_1 : Y(z)Y(zq^{-2})Y(zq^{-2}t^4)^{-1} : +$$
$$+\lambda_2 : Y(z)Y(zq^{-4}t^2)^{-1}Y(zt^2) : +\lambda_3 : Y(z)Y(zq^{-4}t^2)^{-1}Y(zq^{-2}t^4)^{-1} : +$$
$$+ : Y(zq^{-2}t^2)^{-1}Y(zq^{-4}t^2)^{-1}Y(zq^{-2}t^4)^{-1} :,$$
where
$$\lambda_1 = \frac{(q^4t^2-1)(q^2-t^2)}{(q^4-t^2)(q^2t^2-1)}; \quad \lambda_2 = \frac{(q^2t^4-1)(q^2-t^2)}{(q^2-t^4)(q^2t^2-1)}; \quad \lambda_3 = \frac{(q^2+1)(t^2+1)(q^2-t^2)^2}{(q^4-t^2)(q^2-t^4)}.$$

Remark 11. (1) A priori, the algorithm does not end in finitely many steps.

(2) We can wonder if the regularity and genericity of the initial monomial m implies the regularity (resp. genericity) of all the monomials produced by the algorithm. It is not the case as shown by the following example in type D_4 :
Let us consider the following indexation of the Dynkin diagram in type D_4 :



We consider the initial dominant generic regular monomial :

$$m = Y_2(z).$$

Step 1 : The term $Y_2(z)$ is admissible in m. We get the monomial :

$$m_1 = Y_1(zq^{-1}t)Y_2(zq^{-2}t^2)^{-1}Y_3(zq^{-1}t)Y_4(zq^{-1}t).$$

Step 2 : The term $Y_1(zq^{-1}t)$ is admissible in m_1. We get the monomial :

$$m_2 = Y_1(zq^{-3}t^3)^{-1}Y_3(zq^{-1}t)Y_4(zq^{-1}t).$$

Step 3 : The term $Y_3(zq^{-1}t)$ is admissible in m_2. We get the monomial :

$$m_3 = Y_1(zq^{-3}t^3)^{-1}Y_2(zq^{-2}t^2)Y_3(zq^{-3}t^3)^{-1}Y_4(zq^{-1}t).$$

Step 4.a : The term $Y_2(zq^{-2}t^2)$ is admissible in m_3. We get the monomial :

$$m_4 = Y_2(zq^{-4}t^4)^{-1}Y_4(zq^{-1}t)Y_4(zq^{-3}t^3).$$

m_4 is generic but not regular as it contains $Y_4(zq^{-1}t)Y_4(zq^{-3}t^3)$.

Step 4.b : The term $Y_4(zq^{-1}t)$ is admissible in m_3. We get the monomial :

$$m_5 = Y_1(zq^{-3}t^3)^{-1}Y_2(zq^{-2}t^2)^2Y_3(zq^{-3}t^3)^{-1}Y_4(zq^{-3}t^3)^{-1}.$$

m_5 is regular but not generic as it contains $Y_2(zq^{-2}t^2)^2$.

4.2. **Graph representation of the algorithm.** We can represent the algorithm as a graph where the vertices are the monomials appearing in the algorithm and the edges represent the transformations of the algorithm. This definition is inspired by the graph defined by Frenkel and Reshetikhin in [FR99]. However, in this framework, for clarity we omit the coefficients in the graph. But they exist and we can explicitly compute them step by step using the formula 21.

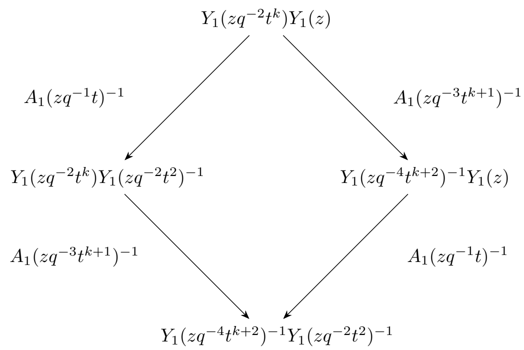
Definition 9. Let m be a generic dominant regular monomial. We assume the algorithm starting from the monomial m never fails. We define the oriented colored graph $G(m)$ associated to the algorithm starting from m as follows :

- The set of vertices of $G(m)$ is the set of monomials $\mathbf{A}(m)$ appearing in the algorithm starting from m.
- There is an oriented edge of color $A_i(zaq^{-r_i}t)^{-1}$ from a vertex m_1 to a vertex m_2 if and only if $Y_i(za)$ is admissible in m_1 and $m_2 = m_1 A_i(zaq^{-r_i}t)^{-1}$.

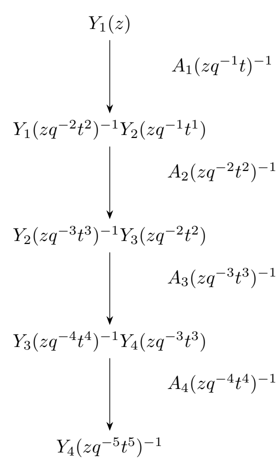
Remark 12. We will draw all the paths such that the edges are directed from up to down so that the upper monomial in the graph $G(m)$ is m.

Example 3. :

- *For $\mathfrak{g} = \mathfrak{sl}_2$, for $k \in \mathbb{Z}\backslash\{0, 2\}$, and $m = Y_1(z)Y_1(zq^{-2}t^k)$ we obtain the following graph*



- *For $\mathfrak{g} = \mathfrak{sl}_4$ and $m = Y_1(z)$:*



4.3. **Well-definedness of the algorithm.** It is not obvious that the algorithm described above is well-defined as the coefficient $\lambda_{i,k}$ defined in (21) seems to depend on the path taken to create the monomial $m'_{i,k}$. For example, in the first example in Example 3, the monomial $Y_1(zq^{-4}t^{k+2})^{-1}Y_1(zq^{-2}t^2)^{-1}$ can be obtained in two different ways. We need to prove that the coefficients obtained are the same.
To prove this, we proceed in two steps. Firstly, we prove in Corollary 2 that if a monomial is reachable via two distinct transformations in directions i and j starting from M and N respectively, then M and N share a common ancestor. This ancestor is guaranteed to exist within the subgraph restricted to the arrows coloured by $A_u(zc)^{-1}$ for $c \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$ and $u \in \{i, j\}$. In order to prove this, we have to prove the technical Lemma 3 which formulates a sufficient (and necessary) condition for a monomial to come from a transformation. Finally, we prove that the coefficient obtained by traversing the cycle along the right-hand path is identical to the one obtained along the left-hand path. This proof is a basic computation in a rank 2 Lie algebra.

After that, we need to prove that if the algorithm does not fail and ends in finitely many steps then the element constructed lies in $\mathbf{W}_{q,t}(\mathfrak{g})$.

Let us begin with some definitions and the technical lemma.

Definition 10. (1) We say that the algorithm *works* if it never fails and ends in finitely many steps.
(2) Let m be a generic dominant regular monomial such that the algorithm starting from m works. For all $X \in \mathbf{A}(m)$, there exists $h \geq 0$, $i_1, \dots i_h \in I$ and $a_1, \dots a_h \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$ such that $X = mA_{i_1}(za_1)^{-1} \cdots A_{i_h}(za_h)^{-1}$. We define the *height* of X to be h. It is well-defined since the variables $A_i(za)$ are algebraically independent in $\mathcal{H}_{q,t}(\mathfrak{g})$ with respect to the normal ordered product.

Lemma 3. Let $m \in \mathbf{M}$ be a dominant regular generic monomial. We assume the algorithm starting from m works. For each monomial $X \in \mathbf{A}(m)$ appearing in the algorithm, for all $i \in I$, for all $a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$, if $d_{i,a}(X) = -1$ and $d_{i,aq^{2r_i}}(X), d_{i,at^{-2}}(X) \neq -1$ then there exists a monomial X' also appearing in the algorithm such that $X' A_i(zaq^{r_i}t^{-1})^{-1} = X$.

Remark 13. This lemma seems natural and seems not so hard to prove. Indeed the proof is not hard but quite technical and long as I did not find a simpler way to do it.

Notation 2. Let $m = Y_{i_1}(za_1)^{\varepsilon_1} \dots Y_{i_k}(za_k)^{\varepsilon_k}$. We will say that a monomial $X \in \mathbf{M}$ *contains* m (or that m *appears* in X) if for all $1 \leq j \leq k$, $d_{i_j,a_j}(X) = \varepsilon_j$.

Proof. We prove it by induction on the height of the monomials.
Height 0 *:* The only monomial with height 0 is the dominant generic monomial from which we start our algorithm. Hence the property at height 0.

Height $h+1$ *:* We assume the property is true for all the monomials with heights $h' \leq h \in \mathbb{N}$. Let us prove it for the monomials with height $h+1$. Let X be a monomial with height $h+1$. Let $i \in I$, $a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$ such that $d_{i,a}(X) = -1$ and $d_{i,aq^{2r_i}}(X), d_{i,at^{-2}}(X) \neq -1$. The monomial appears in the algorithm and its height is strictly positive, so it has to come from a monomial X' with height h. Let $A_j(zbq^{r_j}t^{-1})^{-1}$ be the transformation involved. It implies that $d_{j,b}(X) = -1$ and $d_{j,bq^{2r_j}t^{-2}}(X') = 1$. If $(i,a) = (j,b)$ then we have the result. We assume $(i,a) \neq (j,b)$.

In all this proof we assume the Dynkin subdiagram with nodes $\{i,j\}$ is of type A_1, $A_1 \times A_1$, or A_2. In the other cases the proof has the same structure but the parameters change and the order of transformations differ. The reader can find a proof in type B_2 and G_2 in Appendix B. We have :

$$X = X' A_j(zbq^{r_j}t^{-1})^{-1}.$$

Hence,

$$\begin{cases} d_{j,bq^{2r_j}t^{-2}}(X') = 1 \\ d_{j,b}(X) = -1 \\ d_{j,c}(X) = d_{j,c}(X') \text{ for all } c \notin \{b, bq^{2r_j}t^{-2}\} \\ d_{k,c}(X) \geq d_{k,c}(X') \text{ for all } k \neq j,\ c \in K \end{cases}$$

We know that $d_{i,a}(X) = -1$, then $(i,a) \neq (j, bq^{2r_j}t^{-2})$. Moreover, we know that $(i,a) \neq (j,b)$. By the relations listed above, it implies $d_{i,a}(X') \leq -1$. The monomials are all generic, thus

$$d_{i,a}(X') = -1.$$

Then by the induction assumption, one of the two following assertions is true :

(a) $d_{i,aq^{2r_i}}(X') = -1$ or $d_{i,at^{-2}}(X') = -1$.
(b) There exists a monomial X'' appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X'.$$

a) Firstly we assume (a) is true and X' does contain $Y_i(zaq^{2r_i})^{-1}$ (i.e $d_{i,aq^{2r_i}}(X') = -1$). By assumption, X does not. It implies that the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ simplifies $Y_i(zaq^{2r_i})^{-1}$ and in particular $i \neq j$. Therefore, the expressions depend on the type of the Dynkin diagram generated by the nodes i and j. It is clear that this implies that $C_{i,j} \neq 0$, and the Lie subalgebra generated by the simple roots α_i and α_j is of type A_2, B_2 or G_2.

Let us do it in type A_2. We obtain $b = aqt$. Let $k > 0$ the maximal integer such that $Y_i(zaq^{2s})^{-1}$ appears in X' for all $0 \leq s \leq k$. By definition, $Y_i(zaq^{2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^{2k}t^{-2})^{-1}$ appears in X', then $Y_i(zaq^{2(k-1)})^{-1}Y_i(zaq^{2k}t^{-2})^{-1}$ appears in X'. The monomial has to be regular, so $Y_i(zaq^{2k})^{-1}Y_i(zaq^{2(k-1)}t^{-2})^{-1}$ appears in X'. Thus, X' contains $Y_i(zaq^{2(k-2)})^{-1}Y_i(zaq^{2(k-1)}t^{-2})^{-1}$. We can iterate this reasoning until we get that $Y_i(zat^{-2})^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^{r_j}t^{-1})^{-1} = A_j(zaq^2)^{-1}$. It is absurd by definition of the fields $(A_j(zc))_{c\in\mathbb{C}^*q^{\mathbb{Z}}t^{\mathbb{Z}}}$. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1A_i(zaq^{2k+1}t^{-1})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_sA_i(zaq^{2k-2s+3}t^{-1})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zaq^{2u})^{-1}$ appears in X_s for all $0 \leq u \leq k+1-s$ and by the same argument as before, $Y_i(zaq^{2(u-1)}t^{-2})^{-1}$ does not appear in X_s nor $Y_i(zaq^{2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1}A_i(zaqt^{-1})^{-1} = X_k.$$

Moreover,

$$X_{k+1}A_i(zaqt^{-1})^{-1}A_i(zaq^3t^{-1})^{-1}\cdots A_i(zaq^{2k+1}t^{-1})^{-1} = X'. \tag{22}$$

We recall that $Y_j(zbq^2t^{-2}) = Y_j(zaq^3t^{-1})$ is admissible in X' and $X'A_j(zaq^2)^{-1} = X$. By definition of the algorithm, it implies that $d_{j,aq^3t^{-1}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (22) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aq^3t^{-1}}(X') = d_{j,aq^3t^{-1}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aq^3t^{-1}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$. Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1}\ldots Y_j(zaqt^{-2s-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^3t^{-2u-1})^{-1}$, we know that it contains the monomial $Y_j(zaqt^{-2u+1})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^3t^{-2u+1})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaq^3t^{-1})^{-1}$, which is absurd. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^2t^{-2(s-u)-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{aligned} X_{k+s+2}A_j(zaq^2t^{-2})^{-1}\cdots A_j(zaq^2t^{-2s-2})^{-1}\times \\ A_i(zaqt^{-1})^{-1}\cdots A_i(zaq^{2k+1}t^{-1})^{-1}A_j(zaq^2)^{-1} = X. \end{aligned} \tag{23}$$

We get the following path in the graph representation of the algorithm:

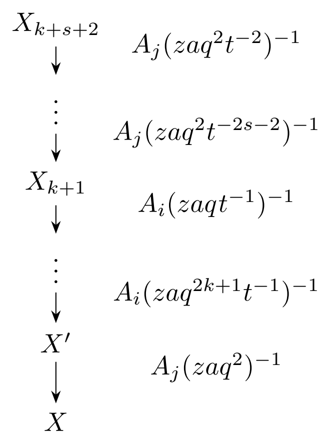


FIGURE 1. One path from X_{k+s+2} to X

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaqt^{-1})^{-1}$ in order to conclude.
Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^4t^{-2})\dots Y_i(zaq^{2k+2}t^{-2})Y_j(zaq^3t^{-3})\dots Y_j(zaq^3t^{-2s-3}). \tag{24}$$

To apply the transformation $A_i(zaq^3t^{-1})^{-1}$, we need to check that $Y_i(zaq^4t^{-2})$ is admissible in X_{k+s+2}.

We know that X_{k+s+2} does not contain $Y_i(zaq^2t^{-2})$ as it would imply that X_{k+s+1} contains $Y_i(zaq^{-2}t^2)^2$ which is absurd as all monomials are generic by assumption. Moreover, if X_{k+s+2} contains $Y_i(zaq^4)$ then the equation (23) implies :

$$d_{i,aq^4}(X_{k+s+2}) = d_{i,aq^4}(X') = 1.$$

However we assumed (a), so $d_{i,aq^2}(X') = -1$. Hence, X' contains $Y_i(zc)Y_i(zcq^{-2})^{-1}$ with $c = aq^4$ and is not regular. It is absurd. Thus, $Y_i(zaq^4t^{-2})$ is admissible in X_{k+s+2} and the algorithm apply the transformation $A_i(zaq^3t^{-1})^{-1}$ to X_{k+s+2}, giving a monomial Z_{k+s+1} such that :

$$X_{k+s+2}A_i(zaq^3t^{-1})^{-1} = Z_{k+s+1}. \tag{25}$$

We assume there exists $1 \le m \le k-1$ such that the algorithm gives Z_{u+s+1} for all $m < u \le k$ such that :

$$\forall m < u \le k, \quad Z_{u+s+2}A_i(zaq^{2(k-u+1)+1}t^{-1})^{-1} = Z_{u+s+1},$$

setting $Z_{k+s+2} = X_{k+s+2}$. We want to prove that the algorithm gives Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{2(k-m+1)+1}t^{-1})^{-1} = Z_{m+s+1}.$$

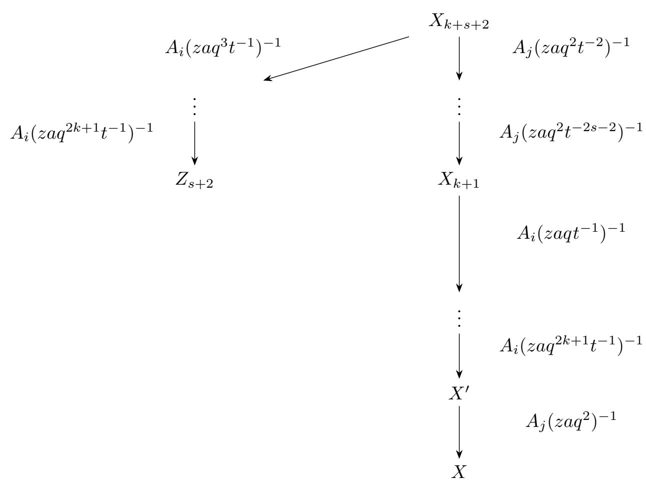
We know that

$$Z_{m+s+2} = X_{k+s+2}A_i(zaq^3t^{-1})^{-1}\cdots A_i(zaq^{2(k-m)+1}t^{-1})^{-1}. \tag{26}$$

Then, Z_{m+s+2} contains $Y_i(zaq^{2(k-m+2)}t^{-2})\dots Y_i(zaq^{2k+2}t^{-2})$.
Let us check that $Y_i(zaq^{2(k-m+2)}t^{-2})$ is admissible in Z_{m+s+2}.
It is clear that Z_{m+s+2} does not contain $Y_i(zaq^{2(k-m+1)}t^{-2})$ as it has been simplified by the transformation $A_i(zaq^{2(k-m)+1}t^{-1})^{-1}$. Moreover, if Z_{m+s+2} contains $Y_i(zaq^{2(k-m+2)})$ then by (26), X_{k+s+2} contains $Y_i(zaq^{2(k-m+2)})$. Thus, according to the Figure 3, X_m contains $Y_i(zaq^{2(k-m+2)})$. However, we read in the same figure that $Y_i(zaq^{2(k-m+2)}t^{-2})$ is admissible in X_m. It is absurd. Hence, Z_{m+s+2} does not contain $Y_i(zaq^{2(k-m+2)})$. Thus, $Y_i(zaq^{2(k-m+2)}t^{-2})$ is admissible in Z_{m+s+2} and the algorithm gives a monomial Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{2(k-m+1)+1}t^{-1})^{-1} = Z_{m+s+1}.$$

By induction, we get monomials $Z_{s+2},\dots,Z_{k+s+1}$ verifiying the equation (26) and leading to the following subgraph :



Furthermore, by (25), we get :

$$d_{j,aq^3t^{-1}}(Z_{s+2}) = d_{j,aq^3t^{-1}}(Z_{k+s+1}) = d_{j,aq^3t^{-1}}(X_{k+s+2}) + 1.$$

But we know by admissibility that we have :

$$d_{j,aq^3t^{-3}}(X_{k+s+2}) = 1, \quad d_{j,aq^3t^{-1}}(Z_{s+2}) = 1, \quad \text{and} \quad d_{j,aq^3t^{-1}}(X_{k+s+2}) = 0.$$

Thus, the monomial Z_{s+2} contains $Y_j(zaq^3t^{-1})\dots Y_j(zaq^3t^{-2s-3})$. We want to check that $Y_j(zaq^3t^{-1})$ is admissible in Z_{s+2}.
If Z_{s+2} contains $Y_j(zaqt^{-1})$ then it contains $Y_j(zaq^3t^{-3})Y_j(zaqt^{-1})$. By regularity, it also contains $Y_j(zaqt^{-3})$. By (26), X_{k+s+2} contains $Y_j(zaqt^{-3})$, and $Y_j(zaq^3t^{-3})$ is not admissible in X_{k+s+2}. It is absurd.
If Z_{s+2} contains $Y_j(zaq^3t)$ then by (26), X_{k+s+2} contains $Y_j(zaq^3t)$. , According to Figure 3, this implies

that X' contains $Y_j(zaq^3t)$. But $Y_j(zaq^3t^{-1})$ is admissible in X'. It is absurd.
Hence, $Y_j(zaq^3t^{-1})$ is admissible in Z_{s+2} and the algorithm gives a monomial Z_{s+1} such that :

$$Z_{s+2}A_j(zaq^2)^{-1} = Z_{s+1}. \tag{27}$$

We assume there exists $0 \leq m \leq s+1$ such that we constructed Z_u for all $m < u \leq s+1$ such that :

$$\forall m < u \leq s+1, \quad Z_{u+1}A_j(zaq^2t^{-2(s+1-u)})^{-1} = Z_u.$$

We want to prove that $Y_j(zaq^3t^{-2(s-m+1)-1})$ is admissible in Z_{m+1} so that the algorithm gives the right monomial Z_m.
It is clear that Z_{m+1} does not contain $Y_j(zaq^3t^{-2(s-m-1)-1})$ as it has been simplified by the last transformation.
Moreover, if Z_{m+1} contains $Y_j(zaqt^{-2(s-m+1)-1})$ then by construction, X_{k+s+2} contains $Y_j(zaqt^{-2(s-m+1)-1})$. According to Figure 3, this implies that X_{k+m+2} contains $Y_j(zaqt^{-2(s-m+1)-1})$. However, we read in the same figure that $Y_j(zaq^3t^{-2(s-m+1)-1})$ is admissible in X_{k+m+2}. It is absurd. Hence, $Y_j(zaq^3t^{-2(s-m+1)-1})$ is admissible in Z_{m+1} and the algorithm gives a monomial Z_m such that :

$$Z_{m+1}A_j(zaq^2t^{-2(s-m+1)})^{-1} = Z_m.$$

Now, we have to prove that $Y_i(zaq^2t^{-2})$ is admissible in Z_0 to prove the result. According to the construcion of the Z_k and to the Figure 3.

We have rigorously the following equality :

$$Z_0 = X_{k+1}A_i(zaq^3t^{-1})^{-1} \dots A_i(zaq^{2k+1}t^{-1})^{-1}A_j(zaq^2)^{-1}.$$

Moreover, $Y_i(zaq^2t^{-2})$ is admissible in X_{k+1}. Hence,

$$d_{i,aq^2t^{-2}}(X_{k+1}) = 1; \quad d_{i,aq^2}(X_{k+1}) = 0; \quad d_{i,at^{-2}}(X_{k+1}) = 0.$$

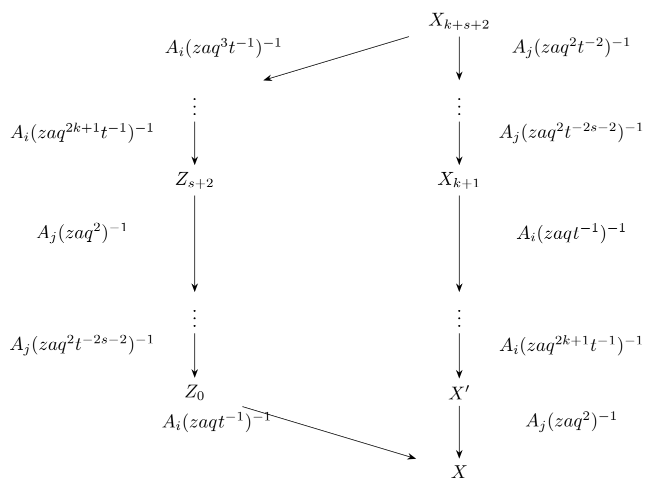
Hence, we get

$$d_{i,aq^2t^{-2}}(Z_0) = 1; \quad d_{i,aq^2}(Z_0) = 0; \quad d_{i,at^{-2}}(Z_0) = 0.$$

Thus, $Y_i(zaq^2t^{-2})$ is admissible in Z_0 and the algorithm gives the transformation :

$$Z_0A_i(zaqt^{-1})^{-1} = X.$$

Hence the result leading to the following graph :



In the case $d_{i,at^{-2}}(X') = -1$, we can do a similar reasoning by exchanging the roles of q^{-1} and t in order to get the same result.
With a similar reasoning we do it in type B_2 and G_2 for (i, j) equals $(1, 2)$ or $(2, 1)$, for $d_{i,aq^{2r_i}}(X') = -1$ and $d_{i,at^{-2}}(X') = -1$ in Appendix B.

b) We assume there exists a monomial X'' also appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X',$$

then X'' does contain $Y_i(zaq^{2r_i}t^{-2})$.

Here there are two cases to consider :

- We have $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_j(zbq^{2r_j}t^{-2})$ comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$.
- X'' contains $Y_j(zbq^{2r_j}t^{-2})$.

Firstly, we assume $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_j(zbq^{2r_j}t^{-2})$ comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$. Hence, $i \neq j$, and we assume the Dynkin subdiagram with nodes $\{i, j\}$ is of type A_2. Thus, $bq^2t^{-2} = aqt^{-1}$, and $bqt^{-1} = a$. Hence,

$$d_{i,a}(A_j(zbqt^{-1})) = 1.$$

However, $d_{i,a}(X) = -1$ and $X'A_j(zaqt^{-1})^{-1} = X$. Thus, $d_{i,a}(X') = -2$ which contradicts the genericity. It is absurd.

Hence, we are in the second case and X'' contains $Y_j(zaq^{2r_j}t^{-2})$.
Let us prove that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i = j$ (so that $r_i = r_j$), then by contradiction we assume that $Y_j(zbq^{2r_j}t^{-2})$ is not admissible in X''. We know that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X' so that the presence of $Y_i(zaq^{2r_i}t^{-2})$ in X'' prevents from doing the transformation $A_j(bq^{r_j}t^{-1})^{-1}$. It implies that $a = bq^{-2r_i}$ or $a = bt^2$. This implies X contains $Y_i(zb)^{-1} = Y_i(zaq^{2r_i})^{-1}$ or $Y_i(zat^{-2})^{-1}$. It is absurd by the initial assumption on X. Thus, $Y_j(zbq^{2r_j}t^{-2})$ is

admissible in X''.
If $i \neq j$ then it is clear that the admissibility of $Y_j(zbq^{2r_j}t^{-2})$ in X' implies its admissibility in X''.
Finally, there exists a monomial

$$X_{new} = X'' A_j(zaq^{r_j}t^{-1})^{-1}$$

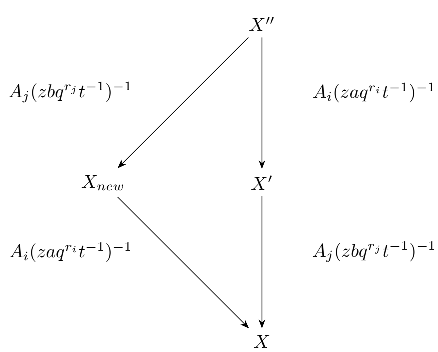
given by the algorithm and X_{new} contains $Y_i(zaq^{2r_i}t^{-2})$. Moreover, if the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$ is not admissible from X_{new} then there exists a factor $Y_i(zat^{-2})$ (resp. $Y_i(zaq^{2r_i})$) blocking this transformation. But this would imply that this factor also appears in X as we have the following equality of monomials :

$$X_{new} = A_i(zaq^{r_i}t^{-1})^{-1}X.$$

We remark that a priori, this equality is not sufficient to get an arrow from X_{new} to X. This equality implies :

$$d_{i,aq^{2r_i}}(X_{new}) = d_{i,aq^{2r_i}}(X),$$

and same for at^{-2}. And hence X does contain $Y_i(zc)$ and $Y_i(zct^2)^{-1}$ with $c = at^{-2}$ (resp. $Y_i(zcq^{-2r_i})^{-1}$ with $c = aq^{2r_i}$) which contradicts the regularity of X. It is impossible. Hence the admissibility of $Y_i(zaq^{2r_i}t^{-2})$ in X_{new}.
Finally we get the following paths :



and hence the result. □

Remark 14. With the result of the Lemma 3 we can prove that the monomials in the algorithm are regular if and only if for all $i \in I$, $a \in K$,

$$d_{i,a}(M) = d_{i,aq^{-2r_i}t^2}(M) \neq 0 \Longrightarrow d_{i,a}(M) = d_{i,aq^{-2r_i}}(M) = d_{i,at^2}(M) = d_{i,aq^{-2r_i}t^2}(M). \qquad (\mathcal{R})$$

Corollary 2. *If a monomial M produced by the algorithm comes for two different transformations :*

$$M = M_0 A_i(za)^{-1} = N_0 A_j(zb)^{-1},$$

then there exists $k \geq 0$ and $2k+1$ monomials N, $M_1, \dots, M_k$ and $N_1, \dots, N_k$ such that:

$$N \to M_k \to \cdots \to M_1 \to M_0 \to M$$

and

$$N \to N_k \to \cdots \to N_1 \to N_0 \to M$$

are both paths produced by the algorithm. Moreover, all the transformations $A_u(zc)^{-1}$ involved satisfy $u \in \{i, j\}$.

Proof. This is proved in the proof of Lemma 3. $\square$

Theorem 1. *a) Each coefficient (21) constructed in the algorithm is well-defined, non-zero, and independent of the path taken. Thus, the algorithm is well-defined.*

b) Given a dominant regular generic monomial m, if the algorithm works (i.e never fails and ends in finitely many steps) then it gives an element $T(m) \in \mathbf{W}_{q,t}(\mathfrak{g})$ which has a unique dominant monomial m with coefficient 1.

Proof. a) There are two things to verify. Firstly, we have to check if the coefficient $\lambda_{i,k}$ defined above is well-defined, and then we have to verify that it is independent of the path chosen.

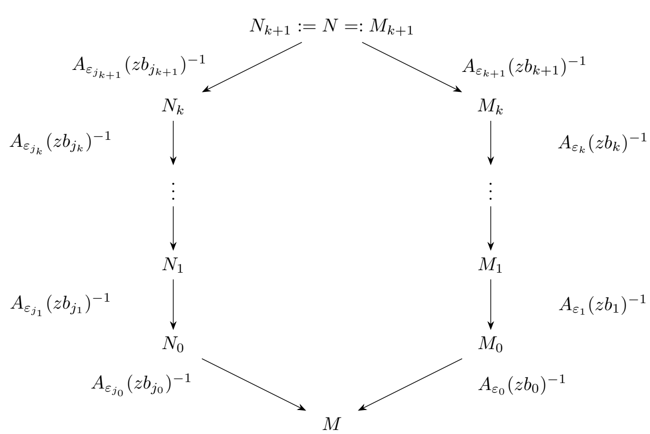
Let $M_1 \to M_2$ be an arrow in the algorithm graph such that $M_2 = M_1 A_1(zaq^{-r_i}t)^{-1}$. We assume

$$M_1 = Y_i(za) \prod_{j=1}^{k} Y_i(zb_j) \prod_{u=1}^{l} Y_i(zc_u)^{-1} M',$$

with $M' \in \mathbb{C}[Y_j(zd)]_{j \neq i, d \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$. The coefficient λ_{M_2} is defined in (21) as a quotient of residues. In section 3.5.3, we get the following explicit formula :

$$\lambda_{M_2} = \lambda_{M_1} \prod_{j=1}^{k} \frac{(1 - q^{-2r_i} ab_j^{-1})(1 - t^2 ab_j^{-1})}{(1 - ab_j^{-1})(1 - q^{-2r_i} t^2 ab_j^{-1})} \prod_{u=1}^{\ell} \frac{(1 - ac_u^{-1})(1 - q^{-2r_i} t^2 ac_u^{-1})}{(1 - q^{-2r_i} ac_u^{-1})(1 - t^2 ac_u^{-1})},$$

where λ_{M_1} is the coefficient associated to the monomial M_1 (see equation (20)).
The Proposition 4 proves that by regularity end genericity the coefficient is well-defined.
Now we have to prove that the coefficient is independent of the choice of the path. We will prove it by induction on the height of monomials (i.e on the length of the paths).
For the first monomial the coefficient is well-defined and equals 1.
Then, we assume all the coefficients above a fixed monomial M are well-defined.
We assume that this monomial M is obtained by a transformation $A_i(zaq^{-r_i}t)^{-1}$ from a monomial N_0 and also by a transformation $A_j(zbq^{-r_j}t)^{-1}$ from a monomial M_0. Hence according to the second point of Corollary 2, there exists $k \in \mathbb{N}$, $b_1, \ldots, b_{k+1} \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$ and monomials N, $M_1, \ldots, M_k$ and $N_1, \ldots, N_k$ such that we have the two paths :



Because of the algebraic independance of the variables $A_i(za)$, $u \mapsto j_u$ is a permutation of $\{0,\dots,k+1\}$. To simplify the notations we write $a_u = b_u q^{r_{\varepsilon_u}} t^{-1}$. For clarity, we will omit all the $Y_u(zc)^\varepsilon$ with $u \notin \{i,j\}$.
Let $\varepsilon'_r, a'_r, \nu'_r$ such that $N = \prod_{u=0}^{k+1} Y_{\varepsilon_u}(za_u) \prod_{r=1}^{d} Y_{\varepsilon'_r}(za'_r)^{\nu_r}$. Here, $Y_{\varepsilon_u}(za_u)$ is admissible in M_u. We can have $a_u = a'_r$ and $\nu_r = -1$, it would mean that $Y_{\varepsilon_u}(za_u)$ does not appear in N but appears after some transformations.
The Corollary 2 ensures that for all $u \in \{0,\dots,k+1\}$, we have $\varepsilon_u \in \{i,j\}$.
Thus, we can consider the Lie subalgebra induced by the nodes $i,j \in I$. This Lie algebra has rank less or equal than 2, then it can be of type A_1, $A_1 \times A_1$, A_2, B_2, or G_2.
Type A_2 : We have

$$M_k = NA_{\varepsilon_{k+1}}(zb_{k+1})^{-1} = NY_{\varepsilon_{k+1}}(za_{k+1})^{-1} Y_{\varepsilon_{k+1}}(za_{k+1}q^{-2r_{\varepsilon_{k+1}}}t^2)^{-1} Y_{\overline{\varepsilon_{k+1}}}(za_{k+1}q^{-r_{\varepsilon_{k+1}}}t)$$

with $\overline{\varepsilon} = i$ if $\varepsilon = j$ and $\overline{\varepsilon} = j$ if $\varepsilon = i$.
Hence, for $u \in [\![-1,k+1]\!]$,

$$M_u = \prod_{r=0}^{u} Y_{\varepsilon_r}(za_r) \prod_{r=u+2}^{k+1} Y_{\varepsilon_r}(za_r q^{-2r_{\varepsilon_r}} t^2)^{-1} \prod_{r=1}^{d} Y_{\varepsilon'_r}(za'_r)^{\nu_r} \prod_{r=u+2}^{k+1} Y_{\overline{\varepsilon_r}}(za_r q^{-r_{\varepsilon_r}} t)$$

Thus,

$$\lambda_{M_u} = \lambda_{M_{u+1}} \prod_{\substack{r=0\\ \varepsilon_{u+1}=\varepsilon_r}}^{u} \frac{(1-a_{u+1}a_r^{-1}q^{-2r_{\varepsilon_r}})(1-a_{u+1}a_r^{-1}t^2)}{(1-a_{u+1}a_r^{-1}q^{-2r_{\varepsilon_r}}t^2)(1-a_{u+1}a_r^{-1})} \prod_{\substack{r=u+2\\ \varepsilon_{u+1}=\varepsilon_r}}^{k} \frac{(1-a_{u+1}a_r^{-1})(1-a_{u+1}a_r^{-1}q^{2r_{\varepsilon_r}}t^{-2})}{(1-a_{u+1}a_r^{-1}t^{-2})(1-a_{u+1}a_r^{-1}q^{2r_{\varepsilon_r}})}$$

$$\times \prod_{\substack{r=0\\ \varepsilon_{u+1}=\varepsilon'_r}}^{s} \left(\frac{(1-a_{u+1}a_r'^{-1}q^{-2r_{\varepsilon'_r}})(1-a_{u+1}a_r'^{-1}t^2)}{(1-a_{u+1}a_r'^{-1}q^{-2r_{\varepsilon'_r}}t^2)(1-a_{u+1}a_r'^{-1})}\right)^{\nu_r} \prod_{\substack{r=u+2\\ \varepsilon_{u+1}\neq\varepsilon_r}}^{k+1} \frac{(1-a_{u+1}a_r^{-1}q^{-r_{\varepsilon_r}}t^{-1})(1-a_{u+1}a_r^{-1}q^{r_{\varepsilon_r}}t)}{(1-a_{u+1}a_r^{-1}q^{-r_{\varepsilon_r}}t)(1-a_{u+1}a_r^{-1}q^{r_{\varepsilon_r}}t^{-1})},$$

This product is well defined by the first point of this proof.
To simplify notations we assume $\lambda_{M_{k+1}} = 1$ and we put γ_u to be this big product :

$$\gamma_u = \lambda_{M_u} \lambda_{M_{u+1}}^{-1}.$$

Then by iterating this computation k times, following the right path, we get

$$\lambda_M = \prod_{u=-1}^{k} \gamma_u$$

$$= \prod_{u=0}^{k+1} \left(\prod_{\substack{r=0\\ \varepsilon_u=\varepsilon_r}}^{u-1} \frac{(1-a_u a_r^{-1}q^{-2r_{\varepsilon_u}})(1-a_u a_r^{-1}t^2)}{(1-a_u a_r^{-1}q^{-2r_{\varepsilon_u}}t^2)(1-a_u a_r^{-1})} \prod_{\substack{r=u+1\\ \varepsilon_u=\varepsilon_r}}^{k} \frac{(1-a_u a_r^{-1})(1-a_u a_r^{-1}q^{2r_{\varepsilon_u}}t^{-2})}{(1-a_u a_r^{-1}t^{-2})(1-a_u a_r^{-1}q^{2r_{\varepsilon_u}})} \right)$$

$$\times \prod_{u=0}^{k+1} \left(\prod_{\substack{r=0\\ \varepsilon_u=\varepsilon'_r}}^{s} \left(\frac{(1-a_u a_r'^{-1}q^{-2r_{\varepsilon_u}})(1-a_u a_r'^{-1}t^2)}{(1-a_u a_r'^{-1}q^{-2r_{\varepsilon_u}}t^2)(1-a_u a_r'^{-1})}\right)^{\nu_r} \prod_{\substack{r=u+1\\ \varepsilon_u\neq\varepsilon_r}}^{k+1} \frac{(1-a_u a_r^{-1}q^{-r_{\varepsilon_u}}t^{-1})(1-a_u a_r^{-1}q^{r_{\varepsilon_u}}t)}{(1-a_u a_r^{-1}q^{-r_{\varepsilon_u}}t)(1-a_u a_r^{-1}q^{r_{\varepsilon_u}}t^{-1})} \right)$$

$$= \prod_{u=0}^{k+1} \left(\prod_{\substack{r=0 \\ \varepsilon_u=\varepsilon_r}}^{u-1} \frac{(1-a_u a_r^{-1} q^{-2r_{\varepsilon_u}})(1-a_u a_r^{-1} t^2)}{(1-a_u a_r^{-1} q^{-2r_{\varepsilon_u}} t^2)(1-a_u a_r^{-1})} \prod_{\substack{r=u+1 \\ \varepsilon_u=\varepsilon_r}}^{k} \frac{(1-a_r a_u^{-1} q^{-2r_{\varepsilon_u}} t^2)(1-a_r a_u^{-1})}{(1-a_r a_u^{-1} q^{-2r_{\varepsilon_u}})(1-a_r a_u^{-1} t^2)} \right)$$

$$\times \prod_{u=0}^{k+1} \prod_{\substack{r=0 \\ \varepsilon_u=\varepsilon'_r}}^{s} \left(\frac{(1-a_u a_r'^{-1} q^{-2r_{\varepsilon_u}})(1-a_u a_r'^{-1} t^2)}{(1-a_u a_r'^{-1} q^{-2r_{\varepsilon_u}} t^2)(1-a_u a_r'^{-1})} \right)^{\nu_r} \prod_{\substack{r=u+1 \\ \varepsilon_u\neq\varepsilon_r}}^{k+1} \frac{(1-a_u a_r^{-1} q^{-r_{\varepsilon_u}} t^{-1})(1-a_u a_r^{-1} q^{r_{\varepsilon_u}} t)}{(1-a_u a_r^{-1} q^{-r_{\varepsilon_u}} t)(1-a_u a_r^{-1} q^{r_{\varepsilon_u}} t^{-1})}$$

$$= \prod_{u=0}^{k+1} \left(\prod_{\substack{r=0 \\ \varepsilon_u=\varepsilon_r}}^{u-1} K_{u,r} \prod_{\substack{r=u+1 \\ \varepsilon_u=\varepsilon_r}}^{k+1} K_{r,u}{}^{-1} \right) \left(\prod_{u=0}^{k+1} C_u \right) \left(\prod_{u=0}^{k+1} \prod_{\substack{r=u+1 \\ \varepsilon_u\neq\varepsilon_r}}^{k+1} K'_{u,r} \right)$$

$$= 1 \times \left(\prod_{u=0}^{k+1} C_u \right) \left(\prod_{\substack{u<r \\ \varepsilon_u\neq\varepsilon_r}} K'_{u,r} \right)$$

with for all $u, r \in [\![0, k+1]\!]$,

$$K_{u,r} := \frac{(1-a_u a_r^{-1} q^{-2r_{\varepsilon_u}})(1-a_u a_r^{-1} t^2)}{(1-a_u a_r^{-1} q^{-2r_{\varepsilon_u}} t^2)(1-a_u a_r^{-1})}, \quad C_u := \prod_{\substack{r=0 \\ \varepsilon_u=\varepsilon'_r}}^{s} \left(\frac{(1-a_u a_r'^{-1} q^{-2r_{\varepsilon_u}})(1-a_u a_r'^{-1} t^2)}{(1-a_u a_r'^{-1} q^{-2r_{\varepsilon_u}} t^2)(1-a_u a_r'^{-1})} \right)^{\nu_r},$$

$$K'_{u,r} := \frac{(1-a_u a_r^{-1} q^{-r_{\varepsilon_u}} t^{-1})(1-a_u a_r^{-1} q^{r_{\varepsilon_u}} t)}{(1-a_u a_r^{-1} q^{-r_{\varepsilon_u}} t)(1-a_u a_r^{-1} q^{r_{\varepsilon_u}} t^{-1})} = K'_{r,u}.$$

Similarly coefficient μ_M computed in the left path is the following :

$$\mu_M = \left(\prod_{u=0}^{k+1} C_{j_u} \right) \left(\prod_{\substack{u<r \\ \varepsilon_u\neq\varepsilon_r}} K'_{j_u, j_r} \right).$$

However, $u \mapsto j_u$ is a bijection. Hence

$$\mu_M = \lambda_M$$

Type A_1 or $A_1 \times A_1$: The proof is exactly the same as for A_2 but without the $K'_{u,r}$.

Type B_2: Let

$$A_i(z) = Y_i(zq^{-1}t) Y_i(zqt^{-1}) Y_j(z)^{-1}$$
$$A_j(z) = Y_j(zq^{-2}t) Y_j(zq^2 t^{-1}) Y_i(zq^{-1})^{-1} Y_i(zq)^{-1}$$
$$(r_i, r_j) = (1, 2)$$

Thus, if $\varepsilon_{k+1} = i$, we get

$$M_k = NA_{\varepsilon_{k+1}}(zb_{k+1})^{-1} = NY_{\varepsilon_{k+1}}(za_{k+1})^{-1} Y_{\varepsilon_{k+1}}(za_{k+1} q^{-2r_{\varepsilon_{k+1}}} t^2)^{-1} Y_{\overline{\varepsilon_{k+1}}}(za_{k+1} q^{-1} t)$$

and if $\varepsilon_{k+1} = j$, we get

$$\begin{aligned} M_k &= NA_{\varepsilon_{k+1}}(zb_{k+1})^{-1} \\ &= NY_{\varepsilon_{k+1}}(za_{k+1})^{-1} Y_{\varepsilon_{k+1}}(za_{k+1} q^{-2r_{\varepsilon_{k+1}}} t^2)^{-1} Y_{\overline{\varepsilon_{k+1}}}(za_{k+1} q^{-1} t) Y_{\overline{\varepsilon_{k+1}}}(za_{k+1} q^{-3} t). \end{aligned}$$

Hence, for $u \in [\![-1, k+1]\!]$,

$$M_u = \prod_{r=1}^{u} Y_{\varepsilon_r}(za_r) \prod_{r=u+2}^{k+1} Y_{\varepsilon_r}(za_r q^{-2r_{\varepsilon_r}} t^2)^{-1} \prod_{r=1}^{d} Y_{\varepsilon'_r}(za'_r)^{\nu_r} \prod_{\substack{r=u+2 \\ \varepsilon_r = i}}^{k+1} Y_{\overline{\varepsilon_r}}(za_r q^{-1} t) \prod_{\substack{r=u+2 \\ \varepsilon_r = j}}^{k+1} Y_{\overline{\varepsilon_r}}(za_r q^{-1} t) Y_{\overline{\varepsilon_r}}(za_r q^{-3} t)$$

By a straightforward computation we get :

$$\lambda_{M_u} = \lambda_{M_{u+1}} \prod_{\substack{r=0 \\ \varepsilon_{u+1} = \varepsilon_r}}^{u} \frac{(1 - a_{u+1} a_r^{-1} q^{-2r_{\varepsilon_{u+1}}})(1 - a_{u+1} a_r^{-1} t^2)}{(1 - a_{u+1} a_r^{-1} q^{-2r_{\varepsilon_{u+1}}} t^2)(1 - a_{u+1} a_r^{-1})} \prod_{\substack{r=u+2 \\ \varepsilon_{u+1} = \varepsilon_r}}^{k} \frac{(1 - a_{u+1} a_r^{-1})(1 - a_{u+1} a_r^{-1} q^{2r_{\varepsilon_{u+1}}} t^{-2})}{(1 - a_{u+1} a_r^{-1} t^{-2})(1 - a_{u+1} a_r^{-1} q^{2r_{\varepsilon_{u+1}}})}$$
$$\times \prod_{\substack{r=0 \\ \varepsilon_{u+1} = \varepsilon'_r}}^{s} \left(\frac{(1 - a_{u+1} a_r'^{-1} q^{-2r_{\varepsilon_{u+1}}})(1 - a_{u+1} a_r'^{-1} t^2)}{(1 - a_{u+1} a_r'^{-1} q^{-2r_{\varepsilon_{u+1}}} t^2)(1 - a_{u+1} a_r'^{-1})} \right)^{\nu_r} \prod_{\substack{r=u+2 \\ \varepsilon_{u+1} \neq \varepsilon_r \\ \varepsilon_r = i}}^{k+1} \frac{(1 - a_{u+1} a_r^{-1} q^{-3} t^{-1})(1 - a_{u+1} a_r^{-1} qt)}{(1 - a_{u+1} a_r^{-1} q^{-3} t)(1 - a_{u+1} a_r^{-1} qt^{-1})}$$
$$\times \prod_{\substack{r=u+2 \\ \varepsilon_{u+1} \neq \varepsilon_r \\ \varepsilon_r = j}}^{k+1} \frac{(1 - a_{u+1} a_r^{-1} q^3 t)(1 - a_{u+1} a_r^{-1} q^{-1} t^{-1})}{(1 - a_{u+1} a_r^{-1} q^3 t^{-1})(1 - a_{u+1} a_r^{-1} q^{-1} t)}$$

Hence,

$$\lambda_M = \left(\prod_{\substack{u<r \\ \varepsilon_u = \varepsilon_r}} K_{u,r} \right) \left(\prod_{\substack{r<u \\ \varepsilon_u = \varepsilon_r}} K_{r,u} \right)^{-1} \left(\prod_{u=0}^{k+1} C_u \right) \left(\prod_{\substack{u,r \\ \varepsilon_r = i \\ \varepsilon_u = j}} K''_{u,r} \right) = \left(\prod_{u=0}^{k+1} C_u \right) \left(\prod_{\substack{u,r \\ \varepsilon_r = i \\ \varepsilon_u = j}} K''_{u,r} \right)$$

with for all $u, r \in [\![0, k+1]\!]$,

$$K_{u,r} := \frac{(1 - a_u a_r^{-1} q^{-2r_{\varepsilon_u}})(1 - a_u a_r^{-1} t^2)}{(1 - a_u a_r^{-1} q^{-2r_{\varepsilon_u}} t^2)(1 - a_u a_r^{-1})}, \quad C_u := \prod_{\substack{r=0 \\ \varepsilon_u = \varepsilon'_r}}^{s} \left(\frac{(1 - a_u a_r'^{-1} q^{-2r_{\varepsilon_u}})(1 - a_u a_r'^{-1} t^2)}{(1 - a_u a_r'^{-1} q^{-2r_{\varepsilon_u}} t^2)(1 - a_u a_r'^{-1})} \right)^{\nu_r},$$

$$K''_{u,r} := \frac{(1 - a_u a_r^{-1} q^{-3} t^{-1})(1 - a_u a_r^{-1} qt)}{(1 - a_u a_r^{-1} q^{-3} t)(1 - a_u a_r^{-1} qt^{-1})},$$

and similarly

$$\mu_M = \left(\prod_{u=0}^{k+1} C_{j_u} \right) \left(\prod_{\substack{\varepsilon_{j_r} = i \\ \varepsilon_{j_u} = j}} K''_{j_u, j_r} \right) = \lambda_M$$

and the result follows in type B_2.

Type G_2: Let

$$A_i(z) = Y_i(zq^{-3}t) Y_i(zq^3 t^{-1}) Y_j(zq^{-2})^{-1} Y_j(z)^{-1} Y_j(zq^2)^{-1}$$
$$A_j(z) = Y_j(zq^{-1}t) Y_j(zqt^{-1}) Y_i(z)^{-1}$$
$$(r_i, r_j) = (3, 1)$$

Thus, if $\varepsilon_{k+1} = i$, we get

$$M_k = N A_{\varepsilon_{k+1}}(zb_{k+1})^{-1}$$

$$= NY_{\varepsilon_{k+1}}(za_{k+1})^{-1}Y_{\varepsilon_{k+1}}(za_{k+1}q^{-2r_{\varepsilon_{k+1}}}t^2)^{-1}Y_{\overline{\varepsilon_{k+1}}}(za_{k+1}q^{-5}t)$$
$$\times Y_{\overline{\varepsilon_{k+1}}}(za_{k+1}q^{-3}t)Y_{\overline{\varepsilon_{k+1}}}(za_{k+1}q^{-1}t)$$

and if $\varepsilon_{k+1} = j$, we get

$$M_k = NA_{\varepsilon_{k+1}}(zb_{k+1})^{-1} = NY_{\varepsilon_{k+1}}(za_{k+1})^{-1}Y_{\varepsilon_{k+1}}(za_{k+1}q^{-2r_{\varepsilon_{k+1}}}t^2)^{-1}Y_{\overline{\varepsilon_{k+1}}}(za_{k+1}q^{-1}t)$$

Hence, for $u \in [\![-1, k+1]\!]$,

$$M_u = \prod_{r=1}^{u} Y_{\varepsilon_r}(za_r) \prod_{r=u+2}^{k+1} Y_{\varepsilon_r}(za_r q^{-2r_{\varepsilon_r}}t^2)^{-1} \prod_{r=1}^{d} Y_{\varepsilon'_r}(za'_r)^{\nu_r} \prod_{\substack{r=u+2\\ \varepsilon_r=j}}^{k+1} Y_{\overline{\varepsilon_r}}(za_r q^{-1}t)$$
$$\times \prod_{\substack{r=u+2\\ \varepsilon_r=i}}^{k+1} Y_{\overline{\varepsilon_r}}(za_r q^{-1}t)Y_{\overline{\varepsilon_r}}(za_r q^{-3}t)Y_{\overline{\varepsilon_r}}(za_r q^{-5}t)$$

Again by a straightforward computation we get :

$$\lambda_{M_u} = \lambda_{M_{u+1}} \prod_{\substack{r=0\\ \varepsilon_{u+1}=\varepsilon_r}}^{u} \frac{(1-a_{u+1}a_r^{-1}q^{-2r_{\varepsilon_{u+1}}})(1-a_{u+1}a_r^{-1}t^2)}{(1-a_{u+1}a_r^{-1}q^{-2r_{\varepsilon_{u+1}}}t^2)(1-a_{u+1}a_r^{-1})} \prod_{\substack{r=u+2\\ \varepsilon_{u+1}=\varepsilon_r}}^{k} \frac{(1-a_{u+1}a_r^{-1})(1-a_{u+1}a_r^{-1}q^{2r_{\varepsilon_{u+1}}}t^{-2})}{(1-a_{u+1}a_r^{-1}t^{-2})(1-a_{u+1}a_r^{-1}q^{2r_{\varepsilon_{u+1}}})}$$
$$\times \prod_{\substack{r=0\\ \varepsilon_{u+1}=\varepsilon'_r}}^{s} \left(\frac{(1-a_{u+1}a_r'^{-1}q^{-2r_{\varepsilon_{u+1}}})(1-a_{u+1}a_r'^{-1}t^2)}{(1-a_{u+1}a_r'^{-1}q^{-2r_{\varepsilon_{u+1}}}t^2)(1-a_{u+1}a_r'^{-1})}\right)^{\nu_r} \prod_{\substack{r=u+2\\ \varepsilon_{u+1}\neq\varepsilon_r\\ \varepsilon_r=i}}^{k+1} \frac{(1-a_{u+1}a_r^{-1}q^{-1}t^{-1})(1-a_{u+1}a_r^{-1}q^5t)}{(1-a_{u+1}a_r^{-1}q^{-1}t)(1-a_{u+1}a_r^{-1}q^5t^{-1})}$$
$$\times \prod_{\substack{r=u+2\\ \varepsilon_{u+1}\neq\varepsilon_r\\ \varepsilon_r=j}}^{k+1} \frac{(1-a_{u+1}a_r^{-1}q^{-5}t^{-1})(1-a_{u+1}a_r^{-1}qt)}{(1-a_{u+1}a_r^{-1}q^{-3}t)(1-a_{u+1}a_r^{-1}qt^{-1})}$$

Hence,

$$\lambda_M = 1 \times \left(\prod_{u=0}^{k+1} C_u\right)\left(\prod_{\substack{\varepsilon_r=i\\ \varepsilon_u=j}} K'''_{u,r}\right)$$

with for all $u, r \in [\![0, k+1]\!]$,

$$C_u := \prod_{\substack{r=0\\ \varepsilon_u=\varepsilon'_r}}^{s} \left(\frac{(1-a_u a_r'^{-1}q^{-2r_{\varepsilon_u}})(1-a_u a_r'^{-1}t^2)}{(1-a_u a_r'^{-1}q^{-2r_{\varepsilon_u}}t^2)(1-a_u a_r'^{-1})}\right)^{\nu_r}, \ K'''_{u,r} := \frac{(1-a_u a_r^{-1}q^5t)(1-a_u a_r^{-1}q^{-1}t^{-1})}{(1-a_u a_r^{-1}q^5t^{-1})(1-a_u a_r^{-1}q^{-1}t)},$$

and similarly

$$\mu_M = \left(\prod_{u=0}^{k+1} C_{j_u}\right)\left(\prod_{\substack{\varepsilon_{j_r}=i\\ \varepsilon_{j_u}=j}} K'''_{j_u,j_r}\right) = \lambda_M,$$

and the result follows in type G_2.

Finally, if n arrows target a monomial M then the coefficient defined from each arrows are pairwise equals. Thus, this coefficient is independent on the path taken and the algorithm is well-defined.

b) Let $\Phi(z) = T(m)$ be a field obtained by the algorithm from a dominant monomial m. We want to prove that for all $i \in I$, $\mathrm{Res}_w[\Phi(z), S_i^-(w)] = 0$.
To prove this, we will prove that all terms of all residues $\mathrm{Res}_w[M, S_i^-(w)]$ for all monomial M appearing in the algorithm cancels each other.
In the section 3.5, we computed the commutator $[M, S_i^-(w)]$ for all generic regular monomial M. Let us recall the result. Let M be a regular generic monomial appearing in the algorithm and let $i \in I$. Let

$$R_M^{(i)} = \{a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}} \mid d_{i,a}(M) = 1, \quad d_{i,aq^{-2r_i}}(M) = 0\},$$

and

$$S_M^{(i)} = \{b \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}} \mid d_{i,b}(M) = -1, \quad d_{i,bq^{2r_i}}(M) = 0\}.$$

Then

$$[M, S_i^-(w)] = \sum_{a \in R_M^{(i)}} \delta\left(\frac{w}{zaq^{-r_i}}\right) C_{a,M} : MS_i^-(w) : + \sum_{b \in S_M^{(i)}} \delta\left(\frac{w}{zbq^{r_i}}\right) D_{b,M} : MS_i^-(w) : \tag{28}$$

Let $a \in R_M^{(i)}$. By definition, $d_{i,a}(M) = 1$ and $d_{i,aq^{-2r_i}}(M) = 0$. Let $s \geq 0$ such that

$$d_{i,a}(M) = d_{i,at^2}(M) = \cdots = d_{i,at^{2s}}(M) = 1, \quad d_{i,at^{2s+2}}(M) = 0.$$

We remark that we do not assume anything on $d_{i,at^{-2}}(M)$. Moreover, if for a $u \in [\![1, s]\!]$, $d_{i,aq^{-2r_i}t^{2u}}(M) = -1$, then M is not regular as $d_i, at^{2u}(M) = 1$. Additionally, if for a $u \in [\![1, s]\!]$, $d_{i,aq^{-2r_i}t^{2u}}(M) = 1$, then

$$d_{i,aq^{-2r_i}t^{2u}}(M) = d_{i,at^{2(u-1)}}(M) = 1.$$

By regularity of M, $d_{i,aq^{-2r_i}t^{2(u-1)}}(M) = 1$. Iterating this argument, we get $d_{i,aq^{-2r_i}}(M) = 1$ which is a contradiction. Hence,

$$d_{i,aq^{-2r_i}}(M) = d_{i,aq^{-2r_i}t^2}(M) = \cdots = d_{i,aq^{-2r_i}t^{2s}}(M) = 0.$$

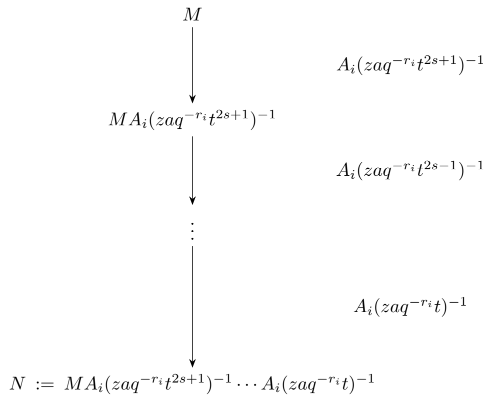
We set

$$M = M' Y_i(za) Y_i(zat^2) \dots Y_i(zat^{2s}) Y_i(za_1) \dots Y_i(za_k) Y_i(zb_1)^{-1} \dots Y_i(zb_n)^{-1},$$

where M' is a monomial in the variables $Y_j(zc)$ for $c \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$, $j \neq i$.
Then, by a straightforward computation, we have

$$C_{a,M} = q^{2(s+1+k-n)r_i}(1 - q^{-2r_i}) \prod_{v=1}^{s} \frac{1 - q^{-2r_i} t^{-2v}}{1 - t^{-2v}} \prod_{j=1}^{k} \frac{1 - q^{-2r_i} a a_j^{-1}}{1 - a a_j^{-1}} \prod_{u=1}^{n} \frac{1 - a b_u^{-1}}{1 - q^{-2r_i} a b_u^{-1}}$$

Moreover, by definition of the algorithm and by construction of s, we get the following admissibility and path of transformations :



Then,

$$N = N'Y_i(zaq^{-2r_i}t^2)^{-1}\ldots Y_i(zaq^{-2r_i}t^{2s+2})^{-1}Y_i(za_1)\ldots Y_i(za_k)Y_i(zb_1)^{-1}\ldots Y_i(zb_n)^{-1}$$

where N' is a monomial in the variables $Y_j(zc)$ for $c \in \mathbb{C}^*q^{\mathbb{Z}}t^{\mathbb{Z}}$, $j \neq i$.
Let us prove that $aq^{-2r_i}t^{2s+2} \in S_N^{(i)}$. Indeed, by definition of the $A_i(zc)$ transformations and by regularity, we have

$$d_{i,aq^{-2r_i}t^{2s+2}}(N) = d_{i,aq^{-2r_i}t^{2s+2}}(M) - 1 = 0 - 1 = -1$$

and

$$d_{i,at^{2s+2}}(N) = d_{i,at^{2s+2}}(M) = 0.$$

Then the delta function $\delta\left(\frac{w}{zaq^{-r_i}t^{2s+2}}\right)$ appears in the commutator $[N, S_i^-(w)]$. Applying $s+1$ difference relations, we get

$$\begin{aligned}
&\delta\left(\frac{w}{zaq^{-r_i}t^{2s+2}}\right) D_{b,N} : NS_i^-(w) : \\
&= \delta\left(\frac{wt^{-(2s+2)}}{zaq^{-r_i}}\right) D_{b,N} : MA_i(zaq^{-r_i}t^{2s+1})^{-1}\cdots A_i(zaq^{-r_i}t)^{-1}S_i^-(w) : \\
&= \delta\left(\frac{wt^{-(2s+2)}}{zaq^{-r_i}}\right) D_{b,N} : MA_i(wt^{-1})^{-1}\cdots A_i(wt^{-2s-1})^{-1}S_i^-(w) : \\
&= q^{2(s+1)r_i}t^{-2(s+1)}\delta\left(\frac{wt^{-(2s+2)}}{zaq^{-r_i}}\right) D_{b,N} : MS_i^-(wt^{-2s-2}) :,
\end{aligned}$$

with $b = aq^{-2r_i}t^{2s+2}$. Again, by a straightforward computation,

$$D_{b,N} = q^{2(k-n-s-1)r_i}(1-q^{2r_i})\prod_{v=0}^{s-1}\frac{1-q^{2r_i}t^{2(s-v)}}{1-t^{2(s-v)}}\prod_{j=1}^{k}\frac{1-q^{-2r_i}t^{2s+2}aa_j^{-1}}{1-t^{2s+2}aa_j^{-1}}\prod_{u=1}^{n}\frac{1-t^{2s+2}ab_u^{-1}}{1-q^{-2r_i}t^{2s+2}ab_u^{-1}}$$

To cancel the residue, we have to verify that

$$0 = \mathrm{Res}_w\left[\lambda_M\delta\left(\frac{w}{zaq^{-r_i}}\right) C_{a,M} : MS_i^-(w) :\right.$$

$$+ \lambda_N q^{2(s+1)r_i} t^{-2(s+1)} \delta\left(\frac{wt^{-(2s+2)}}{zaq^{-r_i}}\right) D_{b,N} : MS_i^-(wt^{-2s-2}) : \Bigg].$$

Thus, the coefficients have to satisfy :

$$\begin{aligned}
\lambda_N &= -\lambda_M q^{-2s-2} \frac{C_{a,M}}{D_{b,N}} \\
&= -\lambda_M q^{2(s+1)r_i} \frac{1-q^{-2r_i}}{1-q^{2r_i}} \prod_{v=1}^{s} \frac{(1-q^{-2r_i}t^{-2v})(1-t^{2v})}{(1-t^{-2v})(1-q^{2r_i}t^{2v})} \times \\
&\times \prod_{j=1}^{k} \frac{(1-q^{-2}aa_j^{-1})(1-t^{2s+2}aa_j^{-1})}{(1-q^{-2}t^{2s+2}aa_j^{-1})(1-aa_j^{-1})} \prod_{u=1}^{n} \frac{(1-q^{-2}t^{2s+2}ab_u^{-1})(1-ab_u^{-1})}{(1-q^{-2}ab_u^{-1})(1-t^{2s+2}ab_u^{-1})} \\
&= \lambda_M \prod_{j=1}^{k} \frac{(1-q^{-2}aa_j^{-1})(1-t^{2s+2}aa_j^{-1})}{(1-q^{-2}t^{2s+2}aa_j^{-1})(1-aa_j^{-1})} \prod_{u=1}^{n} \frac{(1-q^{-2}t^{2s+2}ab_u^{-1})(1-ab_u^{-1})}{(1-q^{-2}ab_u^{-1})(1-t^{2s+2}ab_u^{-1})}
\end{aligned}$$

Applying recursively the formula (21) for each transformation we get :

$$\begin{aligned}
\lambda_N &= \lambda_M q^{-2(s+1)} \prod_{j=0}^{s} \frac{(1-t^{2(j+1)})(1-q^2t^{2(j-s-1)})}{(1-q^{-2}t^{2(j+1)})(1-t^{2(j-s-1)})} \\
&\times \prod_{m=0}^{s} \left[\prod_{j=1}^{k} \frac{(1-q^{-2}t^{2m}aa_j^{-1})(1-t^{2m+2}aa_j^{-1})}{(1-q^{-2}t^{2m+2}aa_j^{-1})(1-t^{2m}ab_j^{-1})} \prod_{u=1}^{n} \frac{(1-q^{-2}t^{2m+2}ab_u^{-1})(1-t^{2m}ab_u^{-1})}{(1-q^{-2}t^{2m}ab_u^{-1})(1-t^{2m+2}ab_u^{-1})}\right] \\
&= \lambda_M \prod_{j=1}^{k} \frac{(1-q^{-2}aa_j^{-1})(1-t^{2s+2}aa_j^{-1})}{(1-q^{-2}t^{2s+2}aa_j^{-1})(1-aa_j^{-1})} \prod_{u=1}^{n} \frac{(1-q^{-2}t^{2s+2}ab_u^{-1})(1-ab_u^{-1})}{(1-q^{-2}ab_u^{-1})(1-t^{2s+2}ab_u^{-1})}.
\end{aligned}$$

Hence, both delta functions cancel each other.

To conclude, we will prove that there is a one to one correspondence between the elements of the two following sets :

$$R := \bigcup_{M\in\mathbf{A}(m)} \left\{(M,i,a) \mid i\in I,\ a\in R_M^{(i)}\right\} \overset{1:1}{\longleftrightarrow} \bigcup_{M\in\mathbf{A}(m)} \left\{(M,i,a) \mid i\in I,\ a\in S_M^{(i)}\right\} =: S,$$

We recall that $\mathbf{A}(m)$ is the set of monomials appearing in the algorithm, or equivalently the set of vertices of the graph $G(m)$.

To lighten the notations, for all $(M,i,a)\in R$, $(N,i,b)\in S$, we denote by $f_{(M,i,a)}(z,w)$ and $g_{(N,i,b)}(z,w)$ the terms of the sum (28). We get:

$$[T(m), : S_i^-(w) :] = \sum_{(M,i,a)\in R} f_{(M,i,a)}(z,w) + \sum_{(N,i,b)\in S} g_{(N,i,b)}(z,w).$$

Let $(M,i,a)\in R$. There exists $s\geq 0$ such that $d_{i,at^{2j}}(M)=1$, and $d_{i,aq^{-2r_i}t^{2j}}(M)=0$ for $0\leq j\leq s$ and $d_{i,at^{2(s+1)}}(M)=0$.

Then as below, let

$$N = MA_i(zaq^{-r_i}t^{2s+1})^{-1} \cdots A_i(zaq^{-r_i}t)^{-1}.$$

We proved that $(N, i, aq^{-2r_i}t^{2s+2}) \in S$, and

$$\mathrm{Res}_w \left[f_{(M,i,a)}(z, w) + g_{(N,i,aq^{-2r_i}t^{2s+2})}(z, w) \right] = 0.$$

To this (M, i, a), we associate $(N, i, aq^{-2r_i}t^{2s+2})$.
Let $(N, i, b) \in S$. There exists $s \geq 0$ such that $d_{i,bt^{-2j}}(N) = -1$ for $0 \leq j \leq s$ and $d_{i,bt^{-2(s+1)}}(N) \geq 0$. Again righ as before, by regularity we can prove that $d_{i,bq^{2r_i}t^{-2j}}(N) = 0$ for $0 \leq j \leq s$.
If $d_{i,bt^{-2(s+1)}}(N) = 1$, then by regularity $d_{i,bt^{-2s}}(N) \neq -1$. It is impossible, thus $d_{i,bt^{-2(s+1)}}(N) = 0$.
By the lemma 3, there exists a path of transformations

$$\begin{array}{c} M := NA_i(zbq^{r_i}t^{-2s-1}) \cdots A_i(zbq^{r_i}t^{-1}) \\ \big\downarrow \; A_i(zbq^{r_i}t^{-1})^{-1} \\ \vdots \\ \big\downarrow \; A_i(zbq^{r_i}t^{-2s+1})^{-1} \\ NA_i(zbq^{r_i}t^{-2s-1}) \\ \big\downarrow \; A_i(zbq^{r_i}t^{-2s-1})^{-1} \\ N \end{array}$$

Then, by construction, $(M, i, bq^{2r_i}t^{-2s-2}) \in R$. Indeed, it is clear by admissibility that $d_{i,bq^{2r_i}t^{-2s-2}}(M) = 1$ and if $d_{i,bt^{-2s-2}}(M) = 1$ then M is not regular as $d_{i,bt^{-2s}}(M) = -1$.
Moreover,

$$d_{i,bq^{2r_i}t^{-2s-2}}(M) = d_{i,bq^{2r_i}t^{-2s}}(M) = \cdots = d_{i,bq^{2r_i}t^{-2}}(M) = 1,$$

and

$$d_{i,bq^{2r_i}}(M) = 0.$$

Hence, the previous construction associate from the tuple $(M, i, bq^{2r_i}t^{-2s-2}) \in R$ the tuple $(N, i, b) \in S$.
Hence,

$$\forall 1 \leq i \leq \ell, \qquad [T(m), : S_i^- :] = 0.$$

Then, we also need to prove that

$$[T(m), : S_i^+ :] = 0.$$

All the formulas are exactly the same for S_i^+ except we replace q^{-r_i} and t. The definition of the coefficient (21) in the algorithm is also symmetric in $q^{-r_i} \leftrightarrow t$. Hence the result.

□

Hence, the algorithm is a valid tool to compute explicitely fields of the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$. As seen in Example 2, the conditions are not too restrictive and the algorithm constructs explicit fields. In the next section, we will apply these results to construct *fundamental* fields of $\mathbf{W}_{q,t}(\mathfrak{g})$ and prove Conjecture 1 in some cases.

5. Fundamental fields in $\mathbf{W}_{q,t}(\mathfrak{g})$

In this section, we give explicit formulas for *fundamental fields* in $\mathbf{W}_{q,t}(\mathfrak{g})$, and we prove Conjecture 1 in new cases. Firstly, we adapt a proof of Frenkel and Reshetikhin [FR98] to our context to prove that the limit $t \to 1$ of any field in $\mathbf{W}_{q,t}(\mathfrak{g})$ is a linear combination of q-characters. Then, we apply the algorithm described in the previous section to construct fundamental fields in $\mathbf{W}_{q,t}(\mathfrak{g})$. There are two possible applications of this algorithm. Firstly, we can basically compute explicitly fields in $\mathbf{W}_{q,t}(\mathfrak{g})$ by applying the algorithm to some dominant monomials. That is what we do for exceptional types. Secondly, we can define intuitively a set of monomials with a unique dominant monomial, and then prove that the set of monomials we introduced is exactly the set of monomials produced by the algorithm starting from this unique dominant monomial. That proves that there exists a linear combination of these monomials lying in $\mathbf{W}_{q,t}(\mathfrak{g})$. Moreover, we know that they specialize to linear combination of q-characters. This allows us to prove Conjecture 1 in some cases. That is what we do for classical types.

Definition 11. A field $\Phi(z) \in \mathbf{W}_{q,t}(\mathfrak{g})$ is called *fundamental* if its expansion has a unique dominant monomial equal to $Y_i(z)$ for some $i \in I$.

For $i = 1$ the result is stated in [FR98] for all classical types (see Section 5.2). In type A_ℓ, for all $(1 \leq i \leq \ell)$, the explicit relation was given by H. Awata, H. Kubo, S. Odake and J. Shiraishi in [AKOS96] and by B. Feigin and E. Frenkel in [FF96]. For $\mathfrak{g}$ of type B_2, C_2, G_2, the result was proved by Bouwknegt and Pilch in [BP98].
Here, we give explicitly the monomials in the fields but we do not compute explicitly the coefficients. We use the algorithm described below to prove that there exists coefficients such that the fields lie in $\mathbf{W}_{q,t}(\mathfrak{g})$. We do it in types A_ℓ $(i \in [\![1,\ell]\!])$, B_ℓ $(i \in [\![1,\ell]\!])$, $C\ell$ $(i \in [\![1,\ell]\!])$, D_ℓ $(i \in \{1,\ell-1,\ell\})$, E_6 $(i \in \{1,5\})$, E_7 $(i = 6)$, F_4 $(i \in \{1,4\})$, G_2 $(i \in \{1,2\})$.
Firstly, we need to prove that the limit $t \to 1$ of any fields in $\mathbf{W}_{q,t}(\mathfrak{g})$ is a linear combination of q-characters.

5.1. The limit $t \to 1$: the q-characters.
In this section, we study the limit $t \to 1$ of the algebra $\mathcal{W}_{q,t}(\mathfrak{g})$. We recall that the parameter t is defined as $t = e^{h\beta}$. The limit $t \to 1$ corresponds to $h\beta \to 0$.

In all this section, we specialize h in $\mathbb{C}\backslash i\pi\mathbb{Q}$. Then q is generic (that is, not a root of unity).
Moreover, we only consider the spectral parameters in $\mathbb{C}^*$.

5.1.1. *The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ and the q-characters.* Drinfeld and Jimbo defined the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ as a q-deformation of the universal enveloping algebra of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ [Dri85, Jim85]. The quantum affine algebra $U_q\hat{\mathfrak{g}}$ has a structure of a Hopf algebra. Thus, the category of its finite-dimensional representations $Rep(U_q\hat{\mathfrak{g}})$ is a monoidal category.
In [CP94, CP95], Chari and Pressley study this category of representations. They prove that the simple representations are characterized by I-tuples of polynomials called *Drinfeld polynomials* with constant coefficient 1. In particular, they study the *fundamental representations* $V_{\omega_i}(a)$, where $i \in I$ and $a \in q^{\mathbb{Z}}$, are the simple representations associated to the I-tuple of polynomials $(P_j(u))_{j\in I}$ such that $P_j(u) = 1$ for $j \neq i$ and $P_i(u) = 1 - au$.
To study $Rep(U_q(\widehat{\mathfrak{g}}))$, Frenkel and Reshetikhin introduced in [FR99], the q-character homomorphism χ_q which is an injective ring homomorphism from the Grothendieck ring of category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ to a polynomial ring in variables $Y_{i,a}$, where $i \in I$ and $a \in q^{\mathbb{Z}}$. They prove that the image

of χ_q is exactly the polynomial subring generated by the q-characters of the fundamental representations $V_{\omega_i}(a)$, where $i \in I$ and $a \in \mathbb{C}^*$.
We first recall the definition of the commutative algebra $\mathcal{Y}$ equipped with screening operators where the q-characters live. Then, we consider the canonical projection of the Heisenberg algebra onto its quotient by $h\beta$, and we define the limit W-algebra as the image of $\mathbf{W}_{q,t}(\mathfrak{g})$ under this projection. Finally, we show that this image is embedded in the kernel of the screening operators.

5.1.2. *The algebra $\mathcal{Y}$ and the screening operators.* Let $\mathcal{Y} = \mathbb{C}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}$ be the polynomial ring in variables $Y_{i,a}$. We define

$$A_{i,a} = Y_{i,aq^{-r_i}} Y_{i,aq^{r_i}} \prod_{j | I_{j,i}=1} Y_{j,a}^{-1} \prod_{j | I_{j,i}=2} Y_{j,aq}^{-1} Y_{j,aq^{-1}}^{-1} \prod_{j | I_{j,i}=3} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1} \in \mathcal{Y}.$$

We define a structure of screening operators on this algebra following the construction in [FR98].
For each $i \in I$, consider the free $\mathcal{Y}$-module generated by symbols $S_{i,x}$ for $x \in \mathbb{C}^*$:

$$\tilde{\mathcal{Y}}_i = \bigoplus_{x \in \mathbb{C}^*} \mathcal{Y} \cdot S_{i,x}$$

Let $\mathcal{Y}_i$ be the quotient of $\tilde{\mathcal{Y}}_i$ by the relations:

$$S_{i,xq^{2r_i}} = A_{i,xq^{r_i}} S_{i,x}, \tag{29}$$

where $A_{i,a}$ is the evaluation of the field $A_i(z)$ defined in Section 3.1 at the spectral parameter a (specifically, the limit value as $\beta \to 0$).
We define a linear operator $\tilde{S}_i : \mathcal{Y} \to \mathcal{Y}_i$ by the formula on the generators:

$$\tilde{S}_i(Y_{j,a}) = \delta_{ij} Y_{i,a} S_{i,a}$$

and extended by the Leibniz rule $\tilde{S}_i(ab) = b\tilde{S}_i(a) + a\tilde{S}_i(b)$. Finally, let $S_i : \mathcal{Y} \to \mathcal{Y}_i$ be the composition of $\tilde{S}_i$ and the projection $\tilde{\mathcal{Y}}_i \to \mathcal{Y}_i$. We call S_i the i-th screening operator. Let χ_q be the q-character homomorphism. Frenkel and Reshetikhin proved the following theorem in [FR99] :

Theorem 2 (Corollary 2, [FR99])**.** *The q-character homomorphism χ_q gives an isomorphism of rings:*

$$\chi_q : Rep(U_q(\hat{\mathfrak{g}})) \xrightarrow{\sim} \mathbb{Z}[T_{i,a}]_{i \in I, a \in \mathbb{C}^*},$$

where $T_{i,a}$ are the q-characters of the fundamental representations of $U_q(\hat{\mathfrak{g}})$

Moreover, E. Frenkel and E. Mukhin proved the following theorem :

Theorem 3 (Theorem 5.1, [FM01])**.** *The image of the q-character homomorphism is equal to the intersection of the kernels of the screening operators:*

$$\mathbb{C} \otimes_{\mathbb{Z}} Im(\chi_q) = \bigcap_{i \in I} Ker(S_i).$$

This gives a generators-description of the intersection of the kernels of the screening operators. The generators are the q-characters of the fundamental representations $T_{i,a}$.

5.1.3. *The limit of* $\mathbf{W}_{q,t}(\mathfrak{g})$. The algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ is over the ring $K = \mathbb{C}[[h,\beta]]$. This allows us to rigorously define the limit as $h\beta \to 0$ by quotienting the coefficient ring by the non-trivial ideal generated by β. In contrast, S_i^- involves negative powers of β, which makes its limit singular in this framework. This asymmetry, however, does not impede our analysis: since the elements considered in the limit are known to commute with S_i^+, this commutativity provides sufficient conditions to show that the limit $\mathbf{W}_{q,1}$ is contained in the intersection of the kernels of the screening operators defined just above.

Let $\mathbf{p} : \mathcal{H}_{q,t}(\mathfrak{g}) \to \mathcal{H}_{q,t}(\mathfrak{g})/\beta\mathcal{H}_{q,t}(\mathfrak{g})$ be the canonical projection onto the quotient of the Heisenberg algebra by the ideal generated by β. Since the commutation relations in $\mathcal{H}_{q,t}(\mathfrak{g})$ contain the factor $(t^n - t^{-n}) = (e^{nh\beta} - e^{-nh\beta})$, which vanishes at $\beta = 0$, the quotient algebra $\mathcal{H}_{q,t}(\mathfrak{g})/\beta\mathcal{H}_{q,t}(\mathfrak{g})$ is commutative. This quotient is isomorphic to the polynomial ring $\mathcal{Y}$ via the identification $Y_i(z) : Y_i(za) \mapsto Y_{i,a}$.
We define the limit $\overline{\mathbf{W}}_{q,1}(\mathfrak{g})$ as the image of the vector subspace $\mathbf{W}_{q,t}(\mathfrak{g})$ under the projection $\mathbf{p}$:

$$\overline{\mathbf{W}}_{q,1}(\mathfrak{g}) := \mathbf{p}\left(\mathbf{W}_{q,t}(\mathfrak{g})\right) \hookrightarrow \mathcal{Y}.$$

Proposition 5. *The limit* $\overline{\mathbf{W}}_{q,1}(\mathfrak{g})$ *is embedded in the intersection of the kernels of the operators* S_i*:*

$$\overline{\mathbf{W}}_{q,1}(\mathfrak{g}) \hookrightarrow \bigcap_{i \in I} Ker(S_i).$$

Proof. The proof is very similar to the reasoning in Paragraph 8.4 in [FR99]. Consider the deformed screening currents $S_i^+(z)$ defined in Section 3.2. We recall that $S_i^+ := S_{i,-1}^+$ is defined as the $(-1)^{th}$ Fourier coefficient of the screening current $S_i^+(z) = \sum_{n\in\mathbb{Z}} S_{i,n}^+ z^{-n}$. In the limit $\beta \to 0$, the family S_i^+ commutes with $\mathcal{H}_{q,t}(\mathfrak{g})$. Hence, for all $x \in \mathcal{H}_{q,t}(\mathfrak{g})$, we have

$$\forall i \in I, \qquad [x, S_i^+] \in \beta\mathcal{H}_{q,t}(\mathfrak{g}).$$

Thus, we define a the following Poisson bracket on $\mathcal{H}_{q,t}(\mathfrak{g})/\beta\mathcal{H}_{q,t}(\mathfrak{g})$ by the formula:

$$\{a, b\} = -\mathbf{p}\left(\frac{1}{2h\beta}[a', b']\right),$$

where a' and b' are any lifts of a and b in $\mathcal{H}_{q,t}(\mathfrak{g})$. This Poisson bracket is well-defined since $\mathcal{H}_{q,t}(\mathfrak{g})$ is free of negative powers of β. This verifies the antisymmetry, the Jacobi identity, and the Leibniz rule for the Poisson bracket.

Moreover, if $a \in \mathbb{C}^*$, $i, j \in I$, and $S_i^+(w)$ the screening current defined in Section 3.2, we have :

$$[Y_i(za), S_j^+(w)] = \delta_{ij}(t^{-2} - 1)\delta\left(\frac{w}{zat}\right) : Y_i(za)S_j^+(w) :,$$

$$\text{Res}_w[Y_i(za), S_j^+(w)] = \text{Res}_w\left[\delta_{ij}(t^{-2} - 1)\delta\left(\frac{w}{zat}\right)Y_i(za)S_j^+(zat)\right],$$

$$[Y_i(za), S_j^+] = \delta_{ij}(t^{-2} - 1)Y_i(za)S_j^+(zat)zat.$$

Hence, in the quotient, we get :

$$\{\overline{Y}_{i,a}, S_j^+\} = \delta_{ij}\overline{Y}_{i,a}\overline{S}_{i,a}.$$

with for all $x \in \mathbb{C}^*$, $i \in I$,

$$\overline{Y}_{i,x} := \mathbf{p}(Y_i(zx)).$$

$$\overline{A}_{i,x} := \mathbf{p}(A_i(zx)).$$
$$\overline{S}_{i,x} := \mathbf{p}(S_i^+(zx)zx).$$

Clearly,

$$\mathcal{H}_{q,t}(\mathfrak{g})/\beta\mathcal{H}_{q,t}(\mathfrak{g}) = K[\overline{Y}_{i,x}^{\pm 1}]_{i\in I, x\in\mathbb{C}^*} =: \overline{\mathcal{Y}},$$

as a commutative algebra. We define a linear operator $\overline{S}_i : \overline{\mathcal{Y}} \to \bigoplus_{i\in I, x\in\mathbb{C}^*} \overline{\mathcal{Y}} \cdot \overline{S}_{i,x}$ by the formula on the generators :

$$\overline{S}_i \cdot \overline{Y}_{j,x} = \delta_{ij}\overline{Y}_{j,x}\overline{S}_{i,x}.$$

Moreover,

$$\begin{aligned}\overline{S}_{i,xq^{-2r_i}} &= \mathbf{p}\left(S_i^+(zxq^{-2r_i})zxq^{-2r_i}\right),\\ &= \mathbf{p}\left(t^{-2}q^{2r_i}A_i(zxq^{-r_i})S_i^+(zx)zxq^{-2r_i}\right),\\ &= \mathbf{p}\left(A_i(zxq^{-r_i})S_i^+(zx)zx\right),\\ &= \overline{A}_{i,xq^{-r_i}}\overline{S}_{i,x}.\end{aligned}$$

Thus, the operator $\overline{S}_i$ satisfies the same relation as the operator S_i defined above. Hence, we can identify $\overline{S}_i$ with S_i and $\overline{\mathcal{Y}}$ with $\mathcal{Y}$, and we get :

$$\mathbb{C}\otimes_{\mathbb{Z}} \operatorname{Im}(\chi_q) = \bigcap_{i\in I} \operatorname{Ker}(S_i).$$

However, for all $\Phi(z) \in \mathbf{W}_{q,t}(\mathfrak{g})$, we have $\operatorname{Res}_w[\Phi(z), S_i^+(w)] = 0$. Under the projection $\mathbf{p}$, the commutator reduces to the action of the derivation S_i on the image $\mathbf{p}(\Phi(z))$. Then, $\mathbf{p}(\Phi(z))$ is in the kernel of S_i for all $i \in I$. This proves the inclusion. □

Remark 15. It is not clear for the moment whether the classical limit $t \to 1$ is equal to the image of the q-character homomorphism. However, Theorem 4 proves that it is the case for a Lie algebra $\mathfrak{g}$ of type A_ℓ, B_ℓ, C_ℓ, or G_2.

It is proved in [FM01] (Theorem 5.1) and [FHR22] (Theorem 3.1) that this intersection is actually isomorphic to the Grothendieck ring of the category of finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$. Under this isomorphism, the unique fields with only one dominant monomial equal to $Y_{i,a}$ correspond to the q-characters of the fundamental representations. Again, in order to know whether the inclusion above is an isomorphism, we need to study the existence of elements in $\mathbf{W}_{q,t}(\mathfrak{g})$ corresponding to these q-characters. This is what we do in the next sections.

5.2. **Explicit formulas for fundamental fields.** To simplify the notations, we introduce the definition of the k-th canonical projection Π_k for all $1 \le k \le \ell$ as follows : Let $\varphi : \mathbf{H}_{q,t}(\mathfrak{g}) \xrightarrow{\sim} K[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$ be the isomorphism defined in Proposition 17. Let

$$\varpi_k : K[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}} \longrightarrow K[Y_{k,a}^{\pm 1}]_{a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}} := K[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}/(Y_{j,a}-1)_{j\neq k, a\in\mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}}$$

be the canonical projection. We define

$$\Pi_k : \varphi^{-1}\circ\varpi_k\circ\varphi : \mathbf{H}_{q,t}(\mathfrak{g}) \longrightarrow \mathbf{H}_{q,t}(\mathfrak{g}).$$

In all this section, we will apply the algorithm described in the previous section to the monomial $Y_i(z)$ for some $i \in I$. Here is the second main theorem of this article:

Theorem 4. *Conjecture 1 holds in types A_ℓ (for all $i \in I$), B_ℓ (for all $i \in I$), C_ℓ (for all $i \in I$), D_ℓ (for $i = 1, \ell-1, \ell$), E_6 (for $i = 1, 5$), E_7 (for $i = 6$), F_4 (for $i = 1, 4$) and G_2 (for $i = 1, 2$).*

It is already known in types A_ℓ and G_2. We give a proof in all other cases.

Proof. In the next sections, we prove that there exists fundamental fields $T_i(z) \in \mathbf{W}_{q,t}(\mathfrak{g})$ whose unique dominant monomial is $Y_i(z)$, appearing with coefficient 1 for all the cases considered in this theorem. Moreover, each of these fields verify that their limit $t \to 1$ also has a unique dominant monomial equal to $Y_i(z)$, appearing with coefficient 1. However, Theorem 3 and Proposition 5 imply that the limit $t \to 1$ of these fields are $\mathbb{C}$-linear combination of q-characters of finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$. Since the q-characters of the fundamental representations are the only ones with a unique dominant monomial equal to $Y_i(z)$, we get that the limit $t \to 1$ of $T_i(z)$ is the q-character of the i-th fundamental representation. This proves that the weights of the fundamental fields $T_i(z)$ are the same as the weights of the i-th fundamental representation, and that the coefficients specialize at $t \to 1$ to positive integers independent of q. Hence the theorem. □

Remark 16. These cases are exactly the cases where the fundamental representations are thin (that is, all their ℓ-weights spaces are of dimension 1). We think their may be a link between the multiplicity of the monomials computed in the Frenkel-Mukhin algorithm and the fact that our algorithm works or not.

Notation 3. To simplify the notations in the next sections, for all $i \in I$, we define the *height* of a tuple $\mathbf{j} = (j_1, \dots, j_i) \in \mathbb{N}^i$ as the integer

$$ht_i(\mathbf{j}) := \sum_{\alpha=1}^{i} (j_\alpha - \alpha).$$

5.2.1. *Type A_ℓ :* In type A_ℓ the algorithm gives all the fundamental fields for $1 \leq i \leq n$. Indeed, it is clear that the fundamental fields $T_i(z)$ are the q-characters of the i-th fundamental representation, with replacing q by $(q^{-1}t)$. The q-characters in type A_n are computed for example in [FR96]. We get :

$$\Lambda_i(z) := Y_i(zq^{-i+1}t^{i+1})Y_{i-1}(zq^{-i}t^i)^{-1}, \qquad i = 1, \dots, \ell+1.$$

The fundamental fields of $\mathbf{W}_{q,t}(A_\ell)$ are

$$T_i(z) = \sum_{1 \leq j_1 < \dots < j_i \leq \ell+1} \Lambda_{j_1}(zq^{-(i-1)}t^{i-1})\Lambda_{j_2}(zq^{-(i-3)}t^{i-3}) \dots \Lambda_{j_i}(zq^{i-1}t^{-(i-1)}),$$

with $\quad i = 1, \dots, \ell+1$. We have: $T_{\ell+1}(z) = 1$.

Proposition 6. [AKOS96, FF96] *For all $1 \leq i \leq \ell$, the field $T_i(z)$ belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$.*

We do the type C_ℓ before the type B_ℓ because there are more intricated conditions on coefficients in type C_ℓ than in type B_ℓ.

5.2.2. *Type C_ℓ :* Let

$$D = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 2 \end{pmatrix}$$

the diagonal matrix such that DC is symmetric.

In type C_ℓ, the algorithm works for all the fundamental fields. Let

$$\Lambda_i(z) := Y_i(zq^{-i+1}t^{i-1})Y_{i-1}(zq^{-i}t^i)^{-1}, \qquad i=1,\dots,\ell.$$

$$\Lambda_{\bar{i}}(z) := Y_{i-1}(zq^{-2\ell+i-2}t^{2\ell-i})Y_i(zq^{-2\ell+i-3}t^{2\ell-i+1})^{-1}, \qquad i=1,\dots,\ell.$$

with $\bar{i} = 2\ell - i + 1$. Then, the fundamental fields of $\mathbf{W}_{q,t}(C_\ell)$ are

$$T_i(z) = \sum_{\mathbf{j}=(j_1,\dots,j_i)\in S} c_{\mathbf{j}}(q,t)\Lambda_{\mathbf{j}}, \tag{30}$$

where

$$\Lambda_{\mathbf{j}} = \Lambda_{j_1}(zq^{-(i-1)}t^{i-1})\Lambda_{j_2}(zq^{-(i-3)}t^{i-3})\dots\Lambda_{j_i}(zq^{i-1}t^{-(i-1)}),$$

with $i = 1,\dots,\ell$, the coefficients $c_{\mathbf{j}}(q,t) \in K$ will be defined below, and S is the set of $1 \le j_1 < \dots < j_i \le 2\ell$ such that if $j_\alpha = k$ and $j_\beta = 2\ell - k + 1$ then $k \ne \ell + \alpha - \beta + 1$ for all $1 \le k \le \ell$.

Proposition 7. *For all $1 \le i \le \ell$, there exists $c_{j_1,\dots,j_i}(q,t) \in K^S$ such that $T_i(z)$ belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$.*

Proof. Let $i \in I$. Firstly, we list all the cases where $d_{k,a}(\Lambda_{\mathbf{j}}) \ne 0$ for a $\mathbf{j} \in S$ and $k \in I$. Firstly, we list the possible cases for $\Pi_k(\Lambda_{\mathbf{j}})$. For each case we construct all the monomials produced by the algorithm from $\Lambda_{\mathbf{j}}$ in direction k and, when possible, we exhibit a monomial $\Lambda_{\mathbf{j}'}$ verifying $ht_i(\mathbf{j}') < ht_i(\mathbf{j})$such that an admissible transformation in $\Lambda_{\mathbf{j}'}$ gives exactly $\Lambda_{\mathbf{j}}$.

a) There exists $1 \le \alpha \le i$ such that $j_\alpha = k$ and for all other $1 \le \beta \le i$, $j_\beta \notin \{k+1, \bar{k}, \overline{k+1}\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-(i+k-2\alpha)}t^{i+k-2\alpha})$. Here, the variable Y_k is admissible and the algorithm constructs exactly the monomial $\Lambda_{\mathbf{j}'}$ which $\mathbf{j}' = (j_1,\dots,j_{\alpha-1}, j_\alpha+1, j_{\alpha+1},\dots,j_i)$, which is in S since $\mathbf{j}$ is and $j_\beta \ne \overline{k+1}$ for all $\beta \ne \alpha$.
We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$.

b) There exists $1 \le \alpha \le i$ such that $j_\alpha = k+1$ and for all other $1 \le \beta \le i$, $j_\beta \notin \{k, \bar{k}, \overline{k+1}\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-(i+k-2\alpha)-2}t^{i+k-2\alpha+2})^{-1}$. Here, the monomial $\Lambda_{\mathbf{j}'}$ is in the case (a), where $\mathbf{j}' = (j_1,\dots,j_{\alpha-1}, j_\alpha-1, j_{\alpha+1},\dots,j_i)$ is in S since $\mathbf{j}$ is and $j_\beta \ne \bar{k}$ for all $\beta \ne \alpha$. Moreover, the transformation $A_k(zq^{-(i+k-2\alpha)-r_k}t^{i+k-2\alpha+1})$ is admissible in $\Lambda_{\mathbf{j}'}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$.

c) There exists $1 \le \alpha \le i$ such that $j_\alpha = \overline{k+1}$, and for all other $1 \le \beta \le i$, $j_\beta \notin \{k, k+1, \bar{k}\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2\ell+k-i+2\alpha-2}t^{2\ell-k+i-2\alpha})$. Here, the variable Y_k is admissible and the algorithm constructs exactly the monomial $\Lambda_{\mathbf{j}'}$ which $\mathbf{j}' = (j_1,\dots,j_{\alpha-1}, j_\alpha+1, j_{\alpha+1},\dots,j_i)$, which is in S since $\mathbf{j}$ is and $j_\beta \ne k$ for all $\beta \ne \alpha$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$.

d) There exists $1 \le \alpha \le i$ such that $j_\alpha = \bar{k}$ and for all other $1 \le \beta \le i$, $j_\beta \notin \{\overline{k+1}, k, k+1\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2\ell+k-i+2\alpha-4}t^{2\ell-k+i-2\alpha+2})^{-1}$. Here, the monomial $\Lambda_{\mathbf{j}'}$ is in the case (c), where $\mathbf{j}' = (j_1,\dots,j_{\alpha-1}, j_\alpha-1, j_{\alpha+1},\dots,j_i)$ is in S since $\mathbf{j}$ is and $j_\beta \ne \bar{k}$ for all $\beta \ne \alpha$. Moreover, the transformation $A_k(zq^{-2\ell+k-i+2\alpha-2-r_k}t^{2\ell-k+i-2\alpha+1})$ is admissible in $\Lambda_{\mathbf{j}'}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$.

e) There exists $1 \le \alpha < \beta \le i$ such that $j_\alpha = k \ne \ell$, $j_\beta = \overline{k+1}$, and for all other $1 \le \gamma \le i$, $j_\gamma \notin \{\bar{k}, k+1\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-(i+k-2\alpha)}t^{i+k-2\alpha})Y_k(zq^{-2\ell+k-i+2\beta-2}t^{2\ell-k+i-2\beta}).$$

In the first spectral parameter q^{-1} and t play symmetric roles and it is not the case in the second term so we are not in the case of a $Y_k(za)Y_k(zaq^{-2}t^2)$. To know which variable Y_k is admissible, we

need to compare the spectral parameters. This is a straightforward computation:
To simplify notations let $a = q^{-(i+k-2\alpha)}t^{i+k-2\alpha}$. We recall q, t are generic variables.

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zaq^{\pm 2r_k}) \iff aq^{\pm 2r_k} = q^{-2\ell+k-i+2\beta-2}t^{2\ell-k+i-2\beta}$$

$$\iff \begin{cases} -(i+k-2\alpha) \pm 2r_k &= -2\ell+k-i+2\beta-2 \\ i+k-2\alpha &= 2\ell-k+i-2\beta \end{cases}$$

$$\iff \begin{cases} \pm 2r_k &= -2 \\ i+k-2\alpha &= 2\ell-k+i-2\beta \end{cases}$$

$$\iff r_k = 1, \quad \Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zaq^{-2}), \quad k = \ell+\alpha-\beta$$

and in the same way,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zat^{\pm 2}) \iff at^{\pm 2} = q^{-2\ell+k-i+2\beta-2}t^{2\ell-k+i-2\beta}$$

$$\iff \begin{cases} -(i+k-2\alpha) &= -2\ell+k-i+2\beta-2 \\ i+k-2\alpha \pm 2 &= 2\ell-k+i-2\beta \end{cases}$$

$$\iff \begin{cases} -(i+k-2\alpha) &= -2\ell+k-i+2\beta-2 \\ \pm 2 &= -2 \end{cases}$$

$$\iff \Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zat^{-2}), \quad k = \ell+\alpha-\beta+1$$

Finally, if $k = \ell+\alpha-\beta$, then the second variable is the only admissible variable and the algorithm constructs the monomial $\Lambda_{\mathbf{j'}}$ with $\mathbf{j'} = (j'_\gamma)_\gamma$ such that $j'_\gamma = j_\gamma$ for $\gamma \notin \{\alpha,\beta\}$ and $(j'_\alpha, j'_\beta) = (k, \overline{k})$. Indeed, $k = \ell+\alpha-\beta \neq \ell+\alpha-\beta+1$ then $\mathbf{j}' \in S$.
If $k = \ell+\alpha-\beta+1$, then the first variable is the only admissible variable and the algorithm constructs the monomial $\Lambda_{\mathbf{j}''}$ with $\mathbf{j}'' = (j''_\gamma)_\gamma$ such that $j''_\gamma = j_\gamma$ for $\gamma \notin \{\alpha,\beta\}$ and $(j''_\alpha, j''_\beta) = (k+1, \overline{k+1})$. Indeed, $k+1 = \ell+\alpha-\beta+2 \neq \ell+\alpha-\beta+1$ then $\mathbf{j}'' \in S$.
Else, the algorithm constructs both monomials, and both $\mathbf{j}'$ and $\mathbf{j}''$ are in S since $k, k+1 \neq \ell+\alpha-\beta+1$. We have $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}') = ht_i(\mathbf{j})+1$.

f) $k \neq \ell$ and there exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k+1$, $j_\beta = \overline{k+1}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{k, \overline{k}\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-(i+k-2\alpha)-2}t^{i+k-2\alpha+2})^{-1}Y_k(zq^{-2\ell+k-i+2\beta-2}t^{2\ell-k+i-2\beta}).$$

This contradicts the regularity condition if and only if $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zat^2)^{-1}$. This equality holds if and only if $k+1 = \ell+\alpha-\beta+1$, a case that does not appear in S.
The second variable is clearly admissible and the algorithm produces a monomial $\Lambda_{\mathbf{j}'}$ with $\mathbf{j}' = (j'_\gamma)_\gamma$ such that $j'_\gamma = j_\gamma$ for $\gamma \notin \{\alpha,\beta\}$ and $(j'_\alpha, j'_\beta) = (k+1, \overline{k})$. Since $\mathbf{j} \in S$, we have $\mathbf{j}' \in S$.
Moreover, let $\mathbf{j}'' = (j''_\gamma)_\gamma$ such that $j''_\gamma = j_\gamma$ for $\gamma \neq \alpha$ and $j''_\alpha = k$. Since $\mathbf{j} \in S$, we have $\mathbf{j}'' \in S$.

Moreover, the transformation $A_k(zq^{-(i+k-2\alpha)-1}t^{i+k-2\alpha+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}''}$ (which is in case (e)) and the algorithm constructs exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$, and $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}) - 1$.

g) There exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k$, $j_\beta = \overline{k}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{\overline{k+1}, k+1\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-(i+k-2\alpha)}t^{i+k-2\alpha})Y_k(zq^{-2\ell+k-i+2\beta-4}t^{2\ell-k+i-2\beta+2})^{-1}.$$

This contradicts the regularity condition if and only if $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zaq^{-2r_k})^{-1}$. This equality holds if and only if $k = \ell + \alpha - \beta + 1$, a case that does not appear in S. The first variable is clearly admissible and the algorithm produces a monomial $\Lambda_{\mathbf{j}'}$ with $\mathbf{j}' = (j'_\gamma)_\gamma$ such that $j'_\gamma = j_\gamma$ for $\gamma \notin \{\alpha, \beta\}$ and $(j'_\alpha, j'_\beta) = (k, \overline{k+1})$. Since $\mathbf{j} \in S$, we have $\mathbf{j}' \in S$.

Moreover, let $\mathbf{j}'' = (j''_\gamma)_\gamma$ such that $j''_\gamma = j_\gamma$ for $\gamma \neq \beta$ and $j''_\beta = \overline{k+1}$. Since $\mathbf{j} \in S$, we have $\mathbf{j}'' \in S$. Moreover, the transformation $A_k(zq^{-2\ell+k-i+2\beta-3}t^{2\ell-k+i-2\beta+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}''}$ (which is in case (e)) and the algorithm constructs exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j})+1$, and $ht_i(\mathbf{j}'') = ht_i(\mathbf{j})-1$.

h) There exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k+1$, $j_\beta = \overline{k}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{\overline{k+1}, k\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-(i+k-2\alpha)-2}t^{i+k-2\alpha+2})^{-1}Y_k(zq^{-2\ell+k-i+2\beta-4}t^{2\ell-k+i-2\beta+2})^{-1}.$$

This contradicts de regularity condition if and only if $r_k = 2$ and $k = \ell$, which is impossible as we would have $j_\alpha = j_\beta$.

Let $\mathbf{j}', \mathbf{j}''$ such that $j'_\gamma = j_\gamma$ for all $\gamma \neq \alpha$, $j'_\alpha = k$ and $j''_\gamma = j_\gamma$ for all $\gamma \neq \beta$, $j''_\beta = \overline{k+1}$. As in the case (e), a straightforward computation shows that if $k \neq \ell + \alpha - \beta$, then $k + 1 \neq \ell + \alpha - \beta + 1$ thus $\mathbf{j}'' \in S$. Moreover, the transformation $A_k(zq^{-2\ell+k-i+2\beta-3}t^{2\ell-k+i-2\beta+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}''}$ (which is in case (g)) and the algorithm constructs exactly $\Lambda_{\mathbf{j}}$.

If $k = \ell+\alpha-\beta$, then $\mathbf{j}' \in S$ and the transformation $A_k(zq^{-2\ell+k-i+2\beta-3}t^{2\ell-k+i-2\beta+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}'}$ (which is in case (f)) and the algorithm constructs exactly $\Lambda_{\mathbf{j}}$.

We have $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$.

We prove by induction that for all $n \in [\![0, i(2\ell - i)]\!]$, the property $\mathcal{P}_n$: " At the n-th step of the algorithm starting from the monomial $Y_i(z)$, we obtain exactly the monomials

$$\Lambda_{j_1}(zq^{-(i-1)}t^{i-1})\Lambda_{j_2}(zq^{-(i-3)}t^{i-3})\dots\Lambda_{j_{i-1}}(zq^{(i-3)}t^{-(i-3)})\Lambda_{j_i}(zq^{(i-1)}t^{-(i-1)}),$$

for all $\mathbf{j} = (j_\alpha)_\alpha$ such that $ht_i(\mathbf{j}) = \sum_{\alpha=1}^{i}(j_\alpha - \alpha) = n$" is true.

If $n = 0$ then we get

$$\Lambda_{j_1}(zq^{-(i-1)}t^{i-1})\Lambda_{j_2}(zq^{-(i-3)}t^{i-3})\dots\Lambda_{j_{i-1}}(zq^{(i-3)}t^{-(i-3)})\Lambda_{j_i}(zq^{(i-1)}t^{-(i-1)}) = Y_i(z)$$

with $j_\alpha = \alpha$ for $\alpha = 1, 2, \dots, i$. Then the only monomial at the 0-th step is $Y_i(z)$.

We assume $\mathcal{P}_n$ is true for an integer $n \in [\![0, i(2\ell - i) - 1]\!]$.

Let us prove that all monomials $\Lambda_{\mathbf{j}}$ such that $ht_i(\mathbf{j}) = n + 1$ are obtained by the algorithm by one transformation from a monomial $\Lambda_{\mathbf{j}'}$ such that $ht_i(\mathbf{j}') = n$. Let $\mathbf{j}$ be such that $ht_i(\mathbf{j}) = n+1 > 0$. The only dominant monomial is for $\mathbf{j} = (1, 2, \dots, i)$, which is obtained at the 0-th step. Hence, there exists a $k \in I$ and $a \in q^{\mathbb{Z}}t^{\mathbb{Z}}$ such that $d_{k,a}(\Lambda_{\mathbf{j}}) < 0$. Hence, we are in one of the cases (b), (d), (f), (g), or (h). In all of these cases, we have exhibited a monomial $\Lambda_{\mathbf{j}'}$ such that $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1 = n$ and an admissible transformation in $\Lambda_{\mathbf{j}'}$ giving exactly $\Lambda_{\mathbf{j}}$, hence $\Lambda_{\mathbf{j}}$ is obtained at the $(n + 1)$-th step of the algorithm.

Conversely, a monomial $X \in \mathbf{M}$ obtained at the $(n+1)$-th step of the algorithm is obtained from a monomial $X' \in \mathbf{M}$ obtained at the n-th step by an admissible transformation. By induction hypothesis, we have $X' = \Lambda_{\mathbf{j}'}$ with $ht_i(\mathbf{j}') = n$. Moreover, by the cases (a), (c), (e), (f), and (g) we have proved that all the monomials produced by the algorithm are of the form $\Lambda_{\mathbf{j}}$ such that $ht_i(\mathbf{j}) = ht_i(\mathbf{j}') + 1 = n + 1$. Hence the result by induction.

□

5.2.3. *Type B_ℓ :* Let

$$D = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

the diagonal matrix such that DC is symmetric.
Let

$$\begin{aligned} \Lambda_i(z) &:= Y_i(zq^{-2i+2}t^{i-1})\, Y_{i-1}(zq^{-2i}t^{i})^{-1}, \qquad i = 1, \dots, \ell-1, \\ \Lambda_\ell(z) &:= Y_\ell(zq^{-2\ell+3}t^{\ell-1})\, Y_\ell(zq^{-2\ell+1}t^{\ell-1})\, Y_{\ell-1}(zq^{-2\ell}t^{\ell})^{-1}, \\ \Lambda_{\ell+1}(z) &:= Y_\ell(zq^{-2\ell+3}t^{\ell-1})\, Y_\ell(zq^{-2\ell-1}t^{\ell+1})^{-1}, \\ \Lambda_{\bar{\ell}}(z) &:= Y_{\ell-1}(zq^{-2\ell+2}t^{\ell})\, Y_\ell(zq^{-2\ell+1}t^{\ell+1})^{-1}\, Y_\ell(zq^{-2\ell-1}t^{\ell+1})^{-1}, \\ \Lambda_{\bar{i}}(z) &:= Y_{i-1}(zq^{-4\ell+2i+2}t^{2\ell-i})\, Y_i(zq^{-4\ell+2i}t^{2\ell-i+1})^{-1}, \qquad i = 1, \dots, \ell-1, \end{aligned}$$

with $\bar{i} = 2\ell + 2 - i$. Then, the fundamental fields of $\mathbf{W}_{q,t}(B_\ell)$ are

$$T_i(z) = \sum_{\mathbf{j}=(j_1,\dots,j_i)\in S} c_{\mathbf{j}}(q,t)\Lambda_{\mathbf{j}}, \tag{31}$$

where

$$\Lambda_{\mathbf{j}} = \Lambda_{j_1}(zq^{-2(i-1)}t^{i-1})\Lambda_{j_2}(zq^{-2(i-3)}t^{i-3})\dots\Lambda_{j_i}(zq^{2(i-1)}t^{-(i-1)}),$$

with $i = 1, \dots, \ell-1$, the coefficients $c_{\mathbf{j}}(q,t) \in K$ will be defined below, and S is the set of $1 \le j_1 \le \cdots \le j_i \le 2\ell+1$ such that for all α, $j_\alpha < j_{\alpha+1}$ or $j_\alpha = j_{\alpha+1} = \ell+1$.
The formula for $T_\ell(z)$, which corresponds to the spinor representation, is inspired by the formula of the ℓ-th fundamental q-character of $\widehat{\mathfrak{so}_{2n+1}}$ in [FR96]:

$$T_\ell(z) = \sum_{\sigma_1,\dots,\sigma_\ell=\pm1} c_{\sigma_1,\dots,\sigma_\ell}(q,t) b_{\sigma_1}(z|1)\, b_{\sigma_2}(zq^{-2\sigma_1}t|2)\cdots b_{\sigma_\ell}(zq^{-2\sigma_1-\cdots-2\sigma_{\ell-1}}t^{\sigma_1+\dots+\sigma_{\ell-1}}|\ell), \tag{32}$$

where

$$\begin{aligned} b_1(z|\ell) &= Y_\ell(zq^{-2\ell}t^{\ell+1})^{-1}, \\ b_1(z|k) &= 1, \qquad k = 1, \dots, \ell-1, \\ b_{-1}(z|\ell) &= Y_{\ell-1}(zq^{-2\ell+1}t^{\ell})^{-1}Y_\ell(zq^{-2\ell+2}t^{\ell-1}), \\ b_{-1}(z|k) &= Y_{k-1}(zq^{-2\ell+1}t^{\ell})^{-1}Y_k(zq^{-2\ell+3}t^{\ell-1}), \qquad k = 1, \dots, \ell-1. \end{aligned}$$

Proposition 8. *For all $1 \le i \le \ell$, there exists $c_{\mathbf{j}}(q,t) \in K^S$ such that $T_i(z)$ belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$.*

Proof. Let $i \in [\![1, \ell-1]\!]$. Firstly, we list all the cases where $d_{k,a}(\Lambda_{\mathbf{j}}) \neq 0$ for a $\mathbf{j} \in S$ and $k \in I$. For each case we construct all the monomials produced by the algorithm from $\Lambda_{\mathbf{j}}$ in direction k and, when possible, we exhibit a monomial $\Lambda_{\mathbf{j}'}$ verifying $ht_i(\mathbf{j}') < ht_i(\mathbf{j})$ such that an admissible transformation in $\Lambda_{\mathbf{j}'}$ gives exactly $\Lambda_{\mathbf{j}}$. However, we need to distinguish the cases $k = \ell$ and $k \neq \ell$. If $k < \ell$:

a) There exists $1 \leq \alpha \leq i$ such that $j_\alpha = k$ and for all other $1 \leq \beta \leq i$, $j_\beta \notin \{k+1, \bar{k}, \overline{k+1}\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2(i+k-2\alpha)}t^{i+k-2\alpha})$.
Here the algorithm produces a monomial $\Lambda_{\mathbf{j}'}$ with $\mathbf{j}' = (j_1, \dots, j_{\alpha-1}, j_\alpha + 1, j_{\alpha+1}, \dots, j_i)$, which is in S. Moreover, $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$.

b) There exists $1 \leq \alpha \leq i$ such that $j_\alpha = k+1$ and for all other $1 \leq \beta \leq i$, $j_\beta \notin \{k, \bar{k}, \overline{k+1}\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2(i+k-2\alpha)-4}t^{i+k-2\alpha+2})^{-1}$. Here, the monomial $\Lambda_{\mathbf{j}'}$ is in the case (a), where $\mathbf{j}' = (j_1, \dots, j_{\alpha-1}, j_\alpha - 1, j_{\alpha+1}, \dots, j_i)$ is in S.
Moreover, the transformation $A_k(zq^{-2(i+k-2\alpha)-2}t^{i+k-2\alpha+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}'}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$.

c) There exists $1 \leq \alpha \leq i$ such that $j_\alpha = \overline{k+1}$, and for all other $1 \leq \beta \leq i$, $j_\beta \notin \{k, k+1, \bar{k}\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-4\ell+2k-2i+4\alpha+2}t^{2\ell-k+i-2\alpha})$.
Here, the variable Y_k is admissible and the algorithm constructs the monomial $\Lambda_{\mathbf{j}'}$ with $\mathbf{j}' = (j_1, \dots, j_{\alpha-1}, j_\alpha + 1, j_{\alpha+1}, \dots, j_i)$, which is in S. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$.

d) There exists $1 \leq \alpha \leq i$ such that $j_\alpha = \bar{k}$ and for all other $1 \leq \beta \leq i$, $j_\beta \notin \{\overline{k+1}, k, k+1\}$. In this case, $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-4\ell+2k-2i+4\alpha-2}t^{2\ell-k+i-2\alpha+2})^{-1}$. Here, the monomial $\Lambda_{\mathbf{j}'}$ is in the case (c), where $\mathbf{j}' = (j_1, \dots, j_{\alpha-1}, j_\alpha - 1, j_{\alpha+1}, \dots, j_i)$ is in S.
Moreover, the transformation $A_k(zq^{-4\ell+2k-2i+4\alpha}t^{2\ell-k+i-2\alpha+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}'}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$.

e) There exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k$, $j_\beta = \overline{k+1}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{\bar{k}, k+1\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2(i+k-2\alpha)}t^{i+k-2\alpha})Y_k(zq^{-4\ell+2k-2i+4\beta+2}t^{2\ell-k+i-2\beta}).$$

In the first spectral parameter $q^{-2} = q^{-r_k}$ and t play symmetric roles and it is not the case in the second term so we are not in the case of a $Y_k(za)Y_k(zaq^{-2r_k}t^2)$. As in type C_ℓ, a straightforward computation is necessary to determine which variable is admissible. However we get that both are admissible. Hence, the algorithm produces two monomials $\Lambda_{\mathbf{j}'}$ and $\Lambda_{\mathbf{j}''}$ with $\mathbf{j}' = (j'_\gamma)_\gamma$ such that $j'_\gamma = j_\gamma$ for $\gamma \notin \{\alpha, \beta\}$ and $(j'_\alpha, j'_\beta) = (k, \bar{k})$, and $\mathbf{j}'' = (j''_\gamma)_\gamma$ such that $j''_\gamma = j_\gamma$ for $\gamma \notin \{\alpha, \beta\}$ and $(j''_\alpha, j''_\beta) = (k+1, \overline{k+1})$. Since $\mathbf{j} \in S$, we have $\mathbf{j}' \in S$ and $\mathbf{j}'' \in S$.
We have $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$.

f) There exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k+1$, $j_\beta = \overline{k+1}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{k, \bar{k}\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2(i+k-2\alpha)-4}t^{i+k-2\alpha+2})^{-1}Y_k(zq^{-4\ell+2k-2i+4\beta+2}t^{2\ell-k+i-2\beta}).$$

A straightforward computation shows that this never contradicts the regularity condition (we never have $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zaq^{-2r_k})$ nor $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zat^2)$). Moreover, the algorithm produces a monomial $\Lambda_{\mathbf{j}'}$ with $\mathbf{j}' = (j_1, \dots, j_{\beta-1}, j_\beta + 1, j_{\beta+1}, \dots, j_i)$, which is in S.
Let $\mathbf{j}'' = (j_1, \dots, j_{\alpha-1}, j_\alpha - 1, j_{\alpha+1}, \dots, j_i)$, which is in S. The monomial $\Lambda_{\mathbf{j}''}$ is in the case (e) and

the transformation $A_k(zq^{-2(i+k-2\alpha)-2}t^{i+k-2\alpha+1})$ is admissible in $\Lambda_{\mathbf{j}''}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$, and $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}) - 1$.

g) There exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k$, $j_\beta = \overline{k}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{\overline{k+1}, k+1\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2(i+k-2\alpha)}t^{i+k-2\alpha})Y_k(zq^{-4\ell+2k-2i+4\beta-2}t^{2\ell-k+i-2\beta+2})^{-1}.$$

A straightforward computation shows that this never contradicts the regularity condition (we never have $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zaq^{-2r_k})$ nor $\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(za)Y_k(zat^2)$). Moreover, the algorithm produces a monomial $\Lambda_{\mathbf{j}'}$ with $\mathbf{j}' = (j_1, \ldots, j_{\alpha-1}, j_\alpha + 1, j_{\alpha+1}, \ldots, j_i)$, which is in S.
Let $\mathbf{j}'' = (j_1, \ldots, j_{\beta-1}, j_\beta - 1, j_{\beta+1}, \ldots, j_i)$, which is in S. The monomial $\Lambda_{\mathbf{j}''}$ is in the case (e) and the transformation $A_k(zq^{-4\ell+2k-2i+4\beta}t^{2\ell-k+i-2\beta+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}''}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$, and $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}) - 1$.

h) There exists $1 \leq \alpha < \beta \leq i$ such that $j_\alpha = k+1$, $j_\beta = \overline{k}$, and for all other $1 \leq \gamma \leq i$, $j_\gamma \notin \{\overline{k+1}, k\}$. In this case,

$$\Pi_k(\Lambda_{\mathbf{j}}) = Y_k(zq^{-2(i+k-2\alpha)-4}t^{i+k-2\alpha+2})^{-1}Y_k(zq^{-4\ell+2k-2i+4\beta-2}t^{2\ell-k+i-2\beta+2})^{-1}.$$

In the first spectral parameter $q^{-2} = q^{-r_k}$ and t play symmetric roles and it is not the case in the second term so we are not in the case of a $Y_k(za)Y_k(zaq^{-2r_k}t^2)$.
Let $\mathbf{j}' = (j_1, \ldots, j_{\alpha-1}, j_\alpha - 1, j_{\alpha+1}, \ldots, j_i)$, which is in S. The monomial $\Lambda_{\mathbf{j}'}$ is in the case (g) and the transformation $A_k(zq^{-2(i+k-2\alpha)-2}t^{i+k-2\alpha+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}'}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$. We could exhibit another monomial such that an admissible transformation gives exactly $\Lambda_{\mathbf{j}}$, but it is not necessary for the proof.

If $k = \ell$, we are in one of the following cases :

i) There exists $1 \leq \alpha \leq i$ and $s \geq 0$ such that $j_\alpha = \ell$, $j_{\alpha+1} = \ldots = j_{\alpha+s} = \ell+1$ and $j_{\alpha+s+1} \notin \{\ell+1, \bar{\ell}\}$. In this case,

$$\Pi_\ell(\Lambda_{\mathbf{j}}) = Y_\ell(zq^{-2\ell-2i+4\alpha-1}t^{\ell+i-2\alpha})Y_\ell(zq^{-2\ell-2i+4\alpha+4s+1}t^{\ell+i-2\alpha-2s}).$$

Hence if $s = 0$ only the first Y_ℓ is admissible. If $s > 0$, both are admissible in the monomial $\Lambda_{\mathbf{j}}$ in the direction ℓ. The algorithm apply these transformations to get $\Lambda_{\mathbf{j}'}$ (resp. $\Lambda_{\mathbf{j}''}$) with $j'_\alpha = j_\alpha + 1 = \ell + 1$ (resp. $j''_{\alpha+s} = j_{\alpha+s} + 1 = \bar{\ell}$), and for all $\gamma \neq \alpha$, (resp. $\gamma \neq \alpha + s$) $j'_\gamma = j_\gamma$ (resp. $j''_\gamma = j_\gamma$). Both $\mathbf{j}'$ and $\mathbf{j}''$ are in S. Moreover, we have $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$.

ii) There exists $1 \leq \alpha \leq i$ and $s \geq 0$ such that $j_\alpha = \ell$, $j_{\alpha+1} = \ldots = j_{\alpha+s} = \ell + 1$ and $j_{\alpha+s+1} = \bar{\ell}$. Here,

$$\Pi_\ell(\Lambda_{\mathbf{j}}) = Y_\ell(zq^{-2\ell-2i+4\alpha-1}t^{\ell+i-2\alpha})Y_\ell(zq^{-2\ell-2i+4\alpha+4s+3}t^{\ell+i-2\alpha-2s})^{-1}.$$

It is clear that it does not contradict the regularity conditoin. Then, there is a unique admissible transformation in the monomial $\Lambda_{\mathbf{j}}$ in the direction k. The algorithm apply this transformation to get $\Lambda_{\mathbf{j}'}$ with $j'_\alpha = j_\alpha + 1 = \ell + 1$ and for all $\gamma \neq \alpha$, $j'_\gamma = j_\gamma$. Let $\mathbf{j}'' = (j''_\gamma)_\gamma \in S$ such that $j''_\gamma = j_\gamma$ for $\gamma \neq \alpha + s + 1$ and $j''_{\alpha+s+1} = j_{\alpha+s+1} - 1 = \ell + 1$. The monomial $\Lambda_{\mathbf{j}''}$ is in the case (i) and the transformation $A_\ell(zq^{-2\ell-2i+4\alpha+4s+4}t^{\ell+i-2\alpha-2s-1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}''}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$, and $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}) - 1$.

iii) There exists $1 \leq \alpha \leq i$ and $s \geq 0$ such that $j_{\alpha-1} \neq \ell$, $j_\alpha = j_{\alpha+1} = \ldots = j_{\alpha+s} = \ell + 1$ and $j_{\alpha+s+1} \neq \ell + 1, \bar{\ell}$. In this case,

$$\Pi_\ell(\Lambda_{\mathbf{j}}) = Y_\ell(zq^{-2\ell-2i+4\alpha+4s+1}t^{\ell+i-2\alpha-2s})Y_\ell(zq^{-2\ell-2i+4\alpha-3}t^{\ell+i-2\alpha+2})^{-1}.$$

Again, it is clear that it does not contradict the regularity condition. There is still a unique admissible transformation in the monomial $\Lambda_{\mathbf{j}}$ in direction k. Hence, the algorithm apply this transformation to get $\Lambda_{\mathbf{j}'}$ with $j'_{\alpha+s} = j_{\alpha+s} + 1 = \bar{\ell}$, and for all $\gamma \neq \alpha + s$, $j'_\gamma = j_\gamma$. Let $\mathbf{j}'' = (j''_\gamma)_\gamma \in S$ such that $j''_\gamma = j_\gamma$ for $\gamma \neq \alpha$ and $j''_\alpha = j_\alpha - 1 = \ell$. The monomial $\Lambda_{\mathbf{j}''}$ is in the case (i) and the transformation $A_\ell(zq^{-2\ell-2i+4\alpha-2}t^{\ell+i-2\alpha-2s+1})^{-1}$ is admissible in $\Lambda_{\mathbf{j}''}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) + 1$, and $ht_i(\mathbf{j}'') = ht_i(\mathbf{j}) - 1$.

iv) There exists $1 \leq \alpha \leq i$ and $s \geq 0$ such that $j_\alpha = \bar{\ell}$, $j_{\alpha-1} = \ldots = j_{\alpha-s} = \ell + 1$ and $j_{\alpha-s-1} < \ell$. Here,

$$\Pi_\ell(\Lambda_{\mathbf{j}}) = Y_\ell(zq^{-2\ell-2i+4\alpha-1}t^{\ell+i-2\alpha+2})^{-1}Y_\ell(zq^{-2\ell-2i+4\alpha-4s-3}t^{\ell+i-2\alpha+2s+2})^{-1}.$$

It is clear that it does not contradict the regularity conditoin. Let $\mathbf{j}' = (j'_\gamma)_\gamma \in S$ such that $j'_\gamma = j_\gamma$ for $\gamma \neq \alpha$ and $j'_\alpha = j_\alpha - 1 = \ell + 1$. The monomial $\Lambda_{\mathbf{j}'}$ is in the case (iii) and the transformation $A_\ell(zq^{-2\ell-2i+4\alpha}t^{\ell+i-2\alpha+1})$ is admissible in $\Lambda_{\mathbf{j}'}$ and gives exactly $\Lambda_{\mathbf{j}}$. We have $ht_i(\mathbf{j}') = ht_i(\mathbf{j}) - 1$. We could exhibit another monomial such that an admissible transformation gives exactly $\Lambda_{\mathbf{j}}$, but it is not necessary for the proof.

Thus, similarly, we can apply the same proof as Proposition 7 and prove by induction that all the monomials obtained by the algorithm at step n are exactly the monomials $\Lambda_{\mathbf{j}}$ such that $ht_i(\mathbf{j}) = n$. Hence, $T_i(z)$ belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$ for all $1 \leq i \leq \ell - 1$.

Now we consider $T_\ell(z)$. Let us prove by induction that for all n, all the monomials obtained at step n of the algorithm are in $T_\ell(z)$. We have $Y_\ell(z) = \prod_k b_{-1}(zq^{2(k-1)}t^{-(k-1)}|k)$. For $\sigma = (\sigma_i)_i \in \{\pm 1\}^\ell$, b_σ will denote the following monomial :

$$b_\sigma := b_{\sigma_1}(z|1)\, b_{\sigma_2}(zq^{-2\sigma_1}t|2) \cdots b_{\sigma_\ell}(zq^{-2\sigma_1-\cdots-2\sigma_{\ell-1}}t^{\sigma_1+\ldots+\sigma_{\ell-1}}|\ell)$$

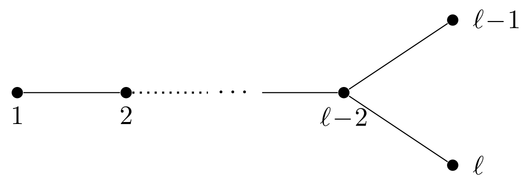
If M is a monomial obtained at step n, if M is anti-dominant then the algorithm stops and there is no step $n+1$. Else, there exists k such that for a $a \in \mathbb{C}^* q^{\mathbb{Z}} t^{\mathbb{Z}}$, $d_{k,a}(M) > 0$. It implies $\sigma_k = -1$ and $\sigma_{k+1} \neq -1$, and $Y_k(za)$ is admissible in b_σ. Let k be such an integer. Then, we define a monomial $b_{\sigma'}$ such that $\sigma'_i = \sigma_i$ if $i \notin \{k, k+1\}$ and $\sigma'_i = -\sigma_i$ for $i \in \{k, k+1\}$ (we take the convention $\sigma_{\ell+1} = 0$). It is easy to see that $b_{\sigma'}$ is the monomial given by the algorithm by taking the expansion in the k^{th} $\mathfrak{sl}_2$ direction in b_σ. Hence all the monomials in the algorithm appear in the sum (32).

Conversely, let $\sigma^{(1)} < \ldots < \sigma^{(2^\ell)}$ be the non-decreasing set of elements of $\{\pm 1\}^\ell$ with respect to the lexicographic order.

We have $\sigma_i^{(1)} = -1$ for all i. Hence $b_{\sigma^{(1)}} = Y_\ell(z)$ appears in the algorithm. Now we assume the monomial $b_{\sigma^{(i)}}$ appears in the algorithm for all $i \leq n$ for a $1 \leq n \leq 2^\ell - 1$. Let us prove that $b_{\sigma^{(n+1)}}$ appears in the algorithm. $\sigma^{(n+1)} > \sigma^{(1)}$ then there exists j such that $\sigma_j^{(n+1)} = 1$. Let $k = \max\{j, \sigma_j^{(n+1)} = 1\}$. By definition of k, $\sigma_{k+1}^{(n+1)} \neq 1$ Let $\sigma \in \{\pm 1\}^\ell$ such that $\sigma_i = \sigma_i^{(n+1)}$ if $i \notin \{k, k+1\}$ and $\sigma_i = -\sigma_i^{(n+1)}$ if $i \in \{k, k+1\}$. Hence there exists $m < n+1$ such that $\sigma = \sigma^{(m)}$. By the induction assumption b_σ appears in the algorithm. Again, it is easy to see that $b_{\sigma^{(n+1)}}$ is the monomial obtained in the algorithm by taking the expansion in the k^{th} $\mathfrak{sl}_2$ direction in b_σ. Hence the result. □

5.2.4. *Type D_ℓ :* The Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix},$$



The fundamental field $T_1(z)$ is given in [FR98] and $T_{\ell-1}(z)$, $T_\ell(z)$ are deduced from the formulas of the associated fundamental q-characters (corresponding to the spinor representations of $\widehat{\mathfrak{so}_{2\ell}}$) in [FR96]. We define the set $S = \{1, \ldots, \ell, \bar{\ell}, \ldots, \bar{1}\}$:

$$\begin{aligned} \Lambda_i(z) &:= Y_i(zq^{-i+1}t^{i-1})\, Y_{i-1}(zq^{-i}t^i)^{-1}, && i = 1, \ldots, \ell-2, \\ \Lambda_{\ell-1}(z) &:= Y_\ell(zq^{-\ell+2}t^{\ell-2})\, Y_{\ell-1}(zq^{-\ell+2}t^{\ell-2})\, Y_{\ell-2}(zq^{-\ell+1}t^{\ell-1})^{-1}, \\ \Lambda_\ell(z) &:= Y_\ell(zq^{-\ell+2}t^{\ell-2})\, Y_{\ell-1}(zq^{-\ell}t^\ell)^{-1}, \\ \Lambda_{\bar{\ell}}(z) &:= Y_{\ell-1}(zq^{-\ell+2}t^{\ell-2})\, Y_\ell(zq^{-\ell}t^\ell)^{-1}, \\ \Lambda_{\overline{\ell-1}}(z) &:= Y_{\ell-2}(zq^{-\ell+1}t^{\ell-1})\, Y_{\ell-1}(zq^{-\ell}t^\ell)^{-1}\, Y_\ell(zq^{-\ell}t^\ell)^{-1}, \\ \Lambda_{\bar{i}}(z) &:= Y_{i-1}(zq^{-2\ell+i+2}t^{2\ell-i-2})\, Y_i(zq^{-2\ell+i+1}t^{2\ell-i-1})^{-1}, && i = 1, \ldots, \ell-2. \end{aligned}$$

Then, we define the first fundamental field as follows:

$$T_1(z) = \sum_{j=1}^{\ell} \Lambda_j(z),$$

with $\quad i = 1, \ldots, \ell$.

We also define the fundamental fields $T_{\ell-1}, T_\ell$ as follows : In these formulas the subscript ε means ℓ if $\varepsilon = +$, and $\ell-1$, if $\varepsilon = -$. Thus, $T_+(z) = T_\ell(z)$, $T_-(z) = T_{\ell-1}(z)$. Now let

$$T_\varepsilon(z) = \sum_{\sigma_j = \pm 1} b_{\sigma_1}(z|1) b_{\sigma_2}(zq^{1-\sigma_1}t^{-1+\sigma_1}|2) \cdots b^{\varepsilon\sigma_1\cdots\sigma_{\ell-1}}_{\sigma_{\ell-1}}(zq^{\ell-2-\sigma_1-\cdots-\sigma_{\ell-2}}t^{-\ell+2+\sigma_1+\cdots+\sigma_{\ell-2}}|\ell-1),$$

where

$$\begin{aligned} b_1^\varepsilon(z|\ell-1) &= Y_\varepsilon(zq^{-\ell}t^\ell)^{-1}, \\ b_1(z|k) &= 1, \qquad k=1,\dots,\ell-2, \\ b_{-1}^\varepsilon(z|\ell-1) &= Y_{\ell-2}(zq^{-\ell+1}t^{\ell-1})^{-1}Y_\varepsilon(zq^{-\ell+2}t^{\ell-2}), \\ b_{-1}(z|k) &= Y_{k-1}(zq^{-\ell+k-1}t^{\ell-k+1})^{-1}Y_k(zq^{-\ell+k}t^{\ell-k}), \qquad k=1,\dots,\ell-2. \end{aligned}$$

Proposition 9. *For all $i \in \{1, \ell-1, \ell\}$, $T_i(z)$ belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$.*

Proof. In [FR98], the authors proved that $T_1(z)$ belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$. The nodes $\ell-1$ and ℓ play symmetric roles then it is sufficient to prove the proposition for $\varepsilon = +$.
Let us prove by induction that for all n, all the monomials obtained at step n of the algorithm are in $T_\ell(z)$. We have

$$Y_\ell(z) = \left(\prod_{k=2}^{\ell-2} b_{-1}(zq^{2(k-1)}t^{-2(k-1)}|k)\right) b_{\sigma_{\ell-1}}^{\varepsilon\sigma_1\cdots\sigma_{\ell-1}}(zq^{\ell-2-\sigma_1-\cdots-\sigma_{\ell-2}}t^{-\ell+2+\sigma_1+\cdots+\sigma_{\ell-2}}|\ell-1)$$

For $\sigma = (\sigma_i)_i \in \{\pm1\}^\ell$, b_σ will denote the following monomial :

$$b_\sigma := b_{\sigma_1}(z|1)b_{\sigma_2}(zq^{1-\sigma_1}t^{-1+\sigma_1}|2)\cdots b_{\sigma_{\ell-1}}^{\varepsilon\sigma_1\cdots\sigma_{\ell-1}}(zq^{\ell-2-\sigma_1-\cdots-\sigma_{\ell-2}}t^{-\ell+2+\sigma_1+\cdots+\sigma_{\ell-2}}|\ell-1)$$

If M is a monomial obtained at step n, if M is anti-dominant then the algorithm stops and there is no step $n+1$. Else, there exists k such that for a $a \in \mathbb{C}^*q^{\mathbb{Z}}t^{\mathbb{Z}}$, $d_{k,a}(M) > 0$. Let $k' = \min(k, \ell-1)$ It implies $\sigma_{k'} = -1$ and $\sigma_{k'+1} \neq -1$, and $Y_k(za)$ is admissible in b_σ. Let k be such an integer. Then, we define a monomial $b_{\sigma'}$ such that $\sigma'_i = \sigma_i$ if $i \notin \{k, k+1\}$ and $\sigma'_i = -\sigma_i$ for $i \in \{k', k'+1\}$ (we take the convention $\sigma_\ell = 0$). It is easy to see that $b_{\sigma'}$ is the monomial given by the algorithm by taking the expansion in the k^{th} $\mathfrak{sl}_2$ direction in b_σ. Hence all the monomials in the algorithm appear in the sum (5.2.4).
Conversely, let $\sigma^{(1)} < \ldots < \sigma^{(2^\ell)}$ be the non-decreasing set of elements of $\{\pm1\}^\ell$ with respect to the lexicographic order.
We have $\sigma_i^{(1)} = -1$ for all i. Hence $b_{\sigma^{(1)}} = Y_\ell(z)$ appears in the algorithm. Now we assume the monomial $b_{\sigma^{(i)}}$ appears in the algorithm for all $i \leq n$ for a $1 \leq n \leq 2^\ell - 1$. Let us prove that $b_{\sigma^{(n+1)}}$ appears in the algorithm. $\sigma^{(n+1)} > \sigma^{(1)}$ then there exists j such that $\sigma_j^{(n+1)} = 1$. Let $k = \max\{j, \sigma_j^{(n+1)} = 1\}$. By definition of k, $\sigma_{k+1}^{(n+1)} \neq 1$. Let $\sigma \in \{\pm1\}^\ell$ such that $\sigma_i = \sigma_i^{(n+1)}$ if $i \notin \{k, k+1\}$ and $\sigma_i = -\sigma_i^{(n+1)}$ if $i \in \{k, k+1\}$. Hence there exists $m < n+1$ such that $\sigma = \sigma^{(m)}$. By the induction assumption b_σ appears in the algorithm. Again, it is easy to see that $b_{\sigma^{(n+1)}}$ is the monomial obtained in the algorithm by taking the expansion in the $(k')^{\text{th}}$ $\mathfrak{sl}_2$ direction in b_σ, with $k' = k$ if $k \leq \ell-2$, else $k' = \ell-1$ (resp. $k' = \ell$) if $\sigma_1^{(n+1)}\sigma_2^{(n+1)}\ldots\sigma_{\ell-1}^{(n+1)} = -1$ (resp $+1$). Hence the result. ∎

5.2.5. *Fundamental fields in exceptional types.* Our algorithm works in type E_6 for $i = 1, 5$, in type E_7 for $i = 6$, in type F_4 for $i = 1, 4$ and in type G_2 for $i = 1, 2$. Moreover, the expressions of the fundamental fields in these cases validate Conjecture 1. Our algorithm fails in all other cases. See the expressions of the fundamental fields in these cases in Appendix A.

6. Conclusive remarks and perspective

Our formal reformulation of the Frenkel and Reshetikhin definition of the deformed W-algebra allowed us to rigourously define an algorithm which provides a practical tool to explicit elements of the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$. Indeed, our proof of Conjecture 1 in various new cases shows a strong example of its efficiency.
We can also remark that our algorithm requires the condition of genericity. However, non-generic fields appear for example in the work of Kimura and Noshita (see Appendix C.2 in [KN26]). Hence, we could remove the genericity condition, which would introduce derivatives of the $Y_i(za)$ variables and slightly change our regularity condition. In this article we did not do this, as it was not necessary for proving Conjecture 1 in the cases considered. We also formulate the following conjecture :

Conjecture 2. *(1) Let $m \in \mathbf{M}$ be a dominant monomial. Then the algorithm starting from m terminates in finitely many steps (with or without a failure).*
(2) Let $m \in \mathbf{M}$ be a dominant regular generic monomial. If the algorithm starting from m creates a non-regular monomial, then there exists no field in $\mathbf{W}_{q,t}(\mathfrak{g})$ with m as its unique dominant monomial.

The first point is verified algorithmically for a large number of dominant monomials m in all types of simple Lie algebra $\mathfrak{g}$.
The second point would imply that Conjecture 1 fails in some cases. For example in type D_4, for $i = 2$, the Remark 11 would prove that a field with $Y_2(z)$ as its unique dominant monomial does not exist in $\mathbf{W}_{q,t}(\mathfrak{g})$.
We believe this second point is true for a technical argument. In the computation of the residue of $[\Phi(z), S_i^{\pm}]$, we can prove that the only way to cancel the residue of a term $[M, S_i^{\pm}]$ is with another term of the form $[M\prod_r A_i(zc_r)^{\varepsilon_r}, S_i^{\pm}]$. However, it seems that the only way for all terms to cancel each other is by following the algorithm described in Section 4.
For instance, we saw that the algorithm fails for all fields for $\mathfrak{g}$ of type E_8. Hence the conjecture would imply that $\mathbf{W}_{q,t}(E_8)$ is the trivial one-dimensional vector space K.
In particular, this would imply that the classical limit $t \to 1$ of $\mathbf{W}_{q,t}(E_8)$ is not equal to the Grothendieck of the finite dimensional representations of the quantum affine algebra $Rep(U_q(\widehat{E_8}))$. However, in [FR98, BP98] or in [FH11, FHR22], it is observed that the deformed W-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ interpolates q-characters and t-(twisted) characters of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ and $U_t({}^L\widehat{\mathfrak{g}})$ in the classical limits. Hence, this framework opens the way to the study of the classical limits of $\mathbf{W}_{q,t}(\mathfrak{g})$. This will be studied in an upcoming paper.

Finally, Remark 16 highlights a link between the deformed W-algebras (or its subset of fields produced by our algorithm) and the thin fundamental representations of quantum affine algebras. This link is still mysterious and deserves to be studied in depth. However, Theorem 4 shows that our algorithm work only for fundamental fields corresponding to fundamental representations which are thin. Moreover, we remark that our algorithm fails for all the non-thin Kirillov-Reshetikhin modules we tried in type C_ℓ, and works for all the thin KR-modules we tried in type A_ℓ, B_ℓ,... Thus, we formulate the following conjecture, adjusting a conjecture of Bouwknegt and Pilch (see Assumption 2.4 in [BP98]):

Conjecture 3. *Our algorithm generates precisely those fields of $\mathbf{W}_{q,t}(\mathfrak{g})$ whose unique dominant monomial, when specialized to $t = 1$, coincides with the dominant monomial of a q-character of a special thin representation of $U_q(\widehat{\mathfrak{g}})$. Here, a special thin representation is defined as one whose q-character possesses a single dominant monomial and has all its coefficients equal to* 1.

Remark 17. This article prove this conjecture for fundamental fields (Theorem 4, and see Section 7.3 in [FH15] for the thinness of fundamental representations).

A. Explicit expressions for fundamental fields in exceptional types

In this section, we list the results of the algorithm computing the monomials and the associated coefficients.

A.1. **Type E.**

A.1.1. *Type E_6.* In type E_6, the algorithm works and gives fundamental fields associated to $i \in \{1,5\}$. T_1, T_5 both have 27 monomials. The algorithm fails for computing $T_i(z)$, $i \neq 1,5$.

There is the list of monomials in $T_1(z)$:

Monomial 1 : $1\cdot : Y_1(z) :$
Monomial 2 : $1\cdot : Y_1(zq^{-2}t^2)^{-1}Y_2(zq^{-1}t^1) :$
Monomial 3 : $1\cdot : Y_2(zq^{-3}t^3)^{-1}Y_3(zq^{-2}t^2) :$
Monomial 4 : $1\cdot : Y_3(zq^{-4}t^4)^{-1}Y_4(zq^{-3}t^3)Y_6(zq^{-3}t^3) :$
Monomial 5 : $1\cdot : Y_4(zq^{-3}t^3)Y_6(zq^{-5}t^5)^{-1} :$
Monomial 6 : $1\cdot : Y_4(zq^{-5}t^5)^{-1}Y_5(zq^{-4}t^4)Y_6(zq^{-3}t^3) :$
Monomial 7 : $1\cdot : Y_3(zq^{-4}t^4)Y_4(zq^{-5}t^5)^{-1}Y_5(zq^{-4}t^4)Y_6(zq^{-5}t^5)^{-1} :$
Monomial 8 : $1\cdot : Y_5(zq^{-6}t^6)^{-1}Y_6(zq^{-3}t^3) :$
Monomial 9 : $1\cdot : Y_2(zq^{-5}t^5)Y_3(zq^{-6}t^6)^{-1}Y_5(zq^{-4}t^4) :$
Monomial 10 : $1\cdot : Y_3(zq^{-4}t^4)Y_5(zq^{-6}t^6)^{-1}Y_6(zq^{-5}t^5)^{-1} :$
Monomial 11 : $1\cdot : Y_2(zq^{-5}t^5)Y_3(zq^{-6}t^6)^{-1}Y_4(zq^{-5}t^5)Y_5(zq^{-6}t^6)^{-1} :$
Monomial 12 : $1\cdot : Y_1(zq^{-6}t^6)Y_2(zq^{-7}t^7)^{-1}Y_5(zq^{-4}t^4) :$
Monomial 13 : $1\cdot : Y_1(zq^{-6}t^6)Y_2(zq^{-7}t^7)^{-1}Y_4(zq^{-5}t^5)Y_5(zq^{-6}t^6)^{-1} :$
Monomial 14 : $1\cdot : Y_2(zq^{-5}t^5)Y_4(zq^{-7}t^7)^{-1} :$
Monomial 15 : $1\cdot : Y_1(zq^{-8}t^8)^{-1}Y_5(zq^{-4}t^4) :$
Monomial 16 : $1\cdot : Y_1(zq^{-6}t^6)Y_2(zq^{-7}t^7)^{-1}Y_3(zq^{-6}t^6)Y_4(zq^{-7}t^7)^{-1} :$
Monomial 17 : $1\cdot : Y_1(zq^{-8}t^8)^{-1}Y_4(zq^{-5}t^5)Y_5(zq^{-6}t^6)^{-1} :$
Monomial 18 : $1\cdot : Y_1(zq^{-8}t^8)^{-1}Y_3(zq^{-6}t^6)Y_4(zq^{-7}t^7)^{-1} :$
Monomial 19 : $1\cdot : Y_1(zq^{-6}t^6)Y_3(zq^{-8}t^8)^{-1}Y_6(zq^{-7}t^7) :$
Monomial 20 : $1\cdot : Y_1(zq^{-8}t^8)^{-1}Y_2(zq^{-7}t^7)Y_3(zq^{-8}t^8)^{-1}Y_6(zq^{-7}t^7) :$
Monomial 21 : $1\cdot : Y_1(zq^{-6}t^6)Y_6(zq^{-9}t^9)^{-1} :$
Monomial 22 : $1\cdot : Y_1(zq^{-8}t^8)^{-1}Y_2(zq^{-7}t^7)Y_6(zq^{-9}t^9)^{-1} :$
Monomial 23 : $1\cdot : Y_2(zq^{-9}t^9)^{-1}Y_6(zq^{-7}t^7) :$
Monomial 24 : $1\cdot : Y_2(zq^{-9}t^9)^{-1}Y_3(zq^{-8}t^8)Y_6(zq^{-9}t^9)^{-1} :$
Monomial 25 : $1\cdot : Y_3(zq^{-10}t^{10})^{-1}Y_4(zq^{-9}t^9) :$
Monomial 26 : $1\cdot : Y_4(zq^{-11}t^{11})^{-1}Y_5(zq^{-10}t^{10}) :$
Monomial 27 : $1\cdot : Y_5(zq^{-12}t^{12})^{-1} :$

There is the list of monomials in $T_5(z)$:

Monomial 1 : $1\cdot : Y_5(z) :$
Monomial 2 : $1\cdot : Y_4(zq^{-1}t^1)Y_5(zq^{-2}t^2)^{-1} :$
Monomial 3 : $1\cdot : Y_3(zq^{-2}t^2)Y_4(zq^{-3}t^3)^{-1} :$
Monomial 4 : $1\cdot : Y_2(zq^{-3}t^3)Y_3(zq^{-4}t^4)^{-1}Y_6(zq^{-3}t^3) :$

Monomial 5 : $1\cdot : Y_2(zq^{-3}t^3)Y_6(zq^{-5}t^5)^{-1} :$
Monomial 6 : $1\cdot : Y_1(zq^{-4}t^4)Y_2(zq^{-5}t^5)^{-1}Y_6(zq^{-3}t^3) :$
Monomial 7 : $1\cdot : Y_1(zq^{-4}t^4)Y_2(zq^{-5}t^5)^{-1}Y_3(zq^{-4}t^4)Y_6(zq^{-5}t^5)^{-1} :$
Monomial 8 : $1\cdot : Y_1(zq^{-6}t^6)^{-1}Y_6(zq^{-3}t^3) :$
Monomial 9 : $1\cdot : Y_1(zq^{-6}t^6)^{-1}Y_3(zq^{-4}t^4)Y_6(zq^{-5}t^5)^{-1} :$
Monomial 10 : $1\cdot : Y_1(zq^{-4}t^4)Y_3(zq^{-6}t^6)^{-1}Y_4(zq^{-5}t^5) :$
Monomial 11 : $1\cdot : Y_1(zq^{-6}t^6)^{-1}Y_2(zq^{-5}t^5)Y_3(zq^{-6}t^6)^{-1}Y_4(zq^{-5}t^5) :$
Monomial 12 : $1\cdot : Y_1(zq^{-4}t^4)Y_4(zq^{-7}t^7)^{-1}Y_5(zq^{-6}t^6) :$
Monomial 13 : $1\cdot : Y_1(zq^{-6}t^6)^{-1}Y_2(zq^{-5}t^5)Y_4(zq^{-7}t^7)^{-1}Y_5(zq^{-6}t^6) :$
Monomial 14 : $1\cdot : Y_2(zq^{-7}t^7)^{-1}Y_4(zq^{-5}t^5) :$
Monomial 15 : $1\cdot : Y_1(zq^{-4}t^4)Y_5(zq^{-8}t^8)^{-1} :$
Monomial 16 : $1\cdot : Y_2(zq^{-7}t^7)^{-1}Y_3(zq^{-6}t^6)Y_4(zq^{-7}t^7)^{-1}Y_5(zq^{-6}t^6) :$
Monomial 17 : $1\cdot : Y_1(zq^{-6}t^6)^{-1}Y_2(zq^{-5}t^5)Y_5(zq^{-8}t^8)^{-1} :$
Monomial 18 : $1\cdot : Y_2(zq^{-7}t^7)^{-1}Y_3(zq^{-6}t^6)Y_5(zq^{-8}t^8)^{-1} :$
Monomial 19 : $1\cdot : Y_3(zq^{-8}t^8)^{-1}Y_5(zq^{-6}t^6)Y_6(zq^{-7}t^7) :$
Monomial 20 : $1\cdot : Y_3(zq^{-8}t^8)^{-1}Y_4(zq^{-7}t^7)Y_5(zq^{-8}t^8)^{-1}Y_6(zq^{-7}t^7) :$
Monomial 21 : $1\cdot : Y_5(zq^{-6}t^6)Y_6(zq^{-9}t^9)^{-1} :$
Monomial 22 : $1\cdot : Y_4(zq^{-7}t^7)Y_5(zq^{-8}t^8)^{-1}Y_6(zq^{-9}t^9)^{-1} :$
Monomial 23 : $1\cdot : Y_4(zq^{-9}t^9)^{-1}Y_6(zq^{-7}t^7) :$
Monomial 24 : $1\cdot : Y_3(zq^{-8}t^8)Y_4(zq^{-9}t^9)^{-1}Y_6(zq^{-9}t^9)^{-1} :$
Monomial 25 : $1\cdot : Y_2(zq^{-9}t^9)Y_3(zq^{-10}t^{10})^{-1} :$
Monomial 26 : $1\cdot : Y_1(zq^{-10}t^{10})Y_2(zq^{-11}t^{11})^{-1} :$
Monomial 27 : $1\cdot : Y_1(zq^{-12}t^{12})^{-1} :$

A.1.2. *Type* E_7. In type E_7, the algorithm works and gives a fundamental field associated to $i = 6$ which contains 56 monomials. The algorithm fails for computing $T_i(z)$, $i \neq 6$.

There is the list of monomials in $T_6(z)$:

Monomial 1 : $1\cdot : Y_6(z) :$
Monomial 2 : $1\cdot : Y_5(zq^{-1}t^1)Y_6(zq^{-2}t^2)^{-1} :$
Monomial 3 : $1\cdot : Y_4(zq^{-2}t^2)Y_5(zq^{-3}t^3)^{-1} :$
Monomial 4 : $1\cdot : Y_3(zq^{-3}t^3)Y_4(zq^{-4}t^4)^{-1} :$
Monomial 5 : $1\cdot : Y_2(zq^{-4}t^4)Y_3(zq^{-5}t^5)^{-1}Y_7(zq^{-4}t^4) :$
Monomial 6 : $1\cdot : Y_2(zq^{-4}t^4)Y_7(zq^{-6}t^6)^{-1} :$
Monomial 7 : $1\cdot : Y_1(zq^{-5}t^5)Y_2(zq^{-6}t^6)^{-1}Y_7(zq^{-4}t^4) :$
Monomial 8 : $1\cdot : Y_1(zq^{-5}t^5)Y_2(zq^{-6}t^6)^{-1}Y_3(zq^{-5}t^5)Y_7(zq^{-6}t^6)^{-1} :$
Monomial 9 : $1\cdot : Y_1(zq^{-7}t^7)^{-1}Y_7(zq^{-4}t^4) :$
Monomial 10 : $1\cdot : Y_1(zq^{-7}t^7)^{-1}Y_3(zq^{-5}t^5)Y_7(zq^{-6}t^6)^{-1} :$
Monomial 11 : $1\cdot : Y_1(zq^{-5}t^5)Y_3(zq^{-7}t^7)^{-1}Y_4(zq^{-6}t^6) :$
Monomial 12 : $1\cdot : Y_1(zq^{-7}t^7)^{-1}Y_2(zq^{-6}t^6)Y_3(zq^{-7}t^7)^{-1}Y_4(zq^{-6}t^6) :$
Monomial 13 : $1\cdot : Y_1(zq^{-5}t^5)Y_4(zq^{-8}t^8)^{-1}Y_5(zq^{-7}t^7) :$
Monomial 14 : $1\cdot : Y_1(zq^{-7}t^7)^{-1}Y_2(zq^{-6}t^6)Y_4(zq^{-8}t^8)^{-1}Y_5(zq^{-7}t^7) :$
Monomial 15 : $1\cdot : Y_2(zq^{-8}t^8)^{-1}Y_4(zq^{-6}t^6) :$
Monomial 16 : $1\cdot : Y_1(zq^{-5}t^5)Y_5(zq^{-9}t^9)^{-1}Y_6(zq^{-8}t^8) :$
Monomial 17 : $1\cdot : Y_2(zq^{-8}t^8)^{-1}Y_3(zq^{-7}t^7)Y_4(zq^{-8}t^8)^{-1}Y_5(zq^{-7}t^7) :$

Monomial 18 : $1\cdot : Y_1(zq^{-7}t^7)^{-1}Y_2(zq^{-6}t^6)Y_5(zq^{-9}t^9)^{-1}Y_6(zq^{-8}t^8) :$
Monomial 19 : $1\cdot : Y_1(zq^{-5}t^5)Y_6(zq^{-10}t^{10})^{-1} :$
Monomial 20 : $1\cdot : Y_2(zq^{-8}t^8)^{-1}Y_3(zq^{-7}t^7)Y_5(zq^{-9}t^9)^{-1}Y_6(zq^{-8}t^8) :$
Monomial 21 : $1\cdot : Y_3(zq^{-9}t^9)^{-1}Y_5(zq^{-7}t^7)Y_7(zq^{-8}t^8) :$
Monomial 22 : $1\cdot : Y_1(zq^{-7}t^7)^{-1}Y_2(zq^{-6}t^6)Y_6(zq^{-10}t^{10})^{-1} :$
Monomial 23 : $1\cdot : Y_3(zq^{-9}t^9)^{-1}Y_4(zq^{-8}t^8)Y_5(zq^{-9}t^9)^{-1}Y_6(zq^{-8}t^8)Y_7(zq^{-8}t^8) :$
Monomial 24 : $1\cdot : Y_2(zq^{-8}t^8)^{-1}Y_3(zq^{-7}t^7)Y_6(zq^{-10}t^{10})^{-1} :$
Monomial 25 : $1\cdot : Y_5(zq^{-7}t^7)Y_7(zq^{-10}t^{10})^{-1} :$
Monomial 26 : $1\cdot : Y_3(zq^{-9}t^9)^{-1}Y_4(zq^{-8}t^8)Y_6(zq^{-10}t^{10})^{-1}Y_7(zq^{-8}t^8) :$
Monomial 27 : $1\cdot : Y_4(zq^{-8}t^8)Y_5(zq^{-9}t^9)^{-1}Y_6(zq^{-8}t^8)Y_7(zq^{-10}t^{10})^{-1} :$
Monomial 28 : $1\cdot : Y_4(zq^{-10}t^{10})^{-1}Y_6(zq^{-8}t^8)Y_7(zq^{-8}t^8) :$
Monomial 29 : $1\cdot : Y_4(zq^{-8}t^8)Y_6(zq^{-10}t^{10})^{-1}Y_7(zq^{-10}t^{10})^{-1} :$
Monomial 30 : $1\cdot : Y_4(zq^{-10}t^{10})^{-1}Y_5(zq^{-9}t^9)Y_6(zq^{-10}t^{10})^{-1}Y_7(zq^{-8}t^8) :$
Monomial 31 : $1\cdot : Y_3(zq^{-9}t^9)Y_4(zq^{-10}t^{10})^{-1}Y_6(zq^{-8}t^8)Y_7(zq^{-10}t^{10})^{-1} :$
Monomial 32 : $1\cdot : Y_3(zq^{-9}t^9)Y_4(zq^{-10}t^{10})^{-1}Y_5(zq^{-9}t^9)Y_6(zq^{-10}t^{10})^{-1}Y_7(zq^{-10}t^{10})^{-1} :$
Monomial 33 : $1\cdot : Y_5(zq^{-11}t^{11})^{-1}Y_7(zq^{-8}t^8) :$
Monomial 34 : $1\cdot : Y_2(zq^{-10}t^{10})Y_3(zq^{-11}t^{11})^{-1}Y_6(zq^{-8}t^8) :$
Monomial 35 : $1\cdot : Y_2(zq^{-10}t^{10})Y_3(zq^{-11}t^{11})^{-1}Y_5(zq^{-9}t^9)Y_6(zq^{-10}t^{10})^{-1} :$
Monomial 36 : $1\cdot : Y_3(zq^{-9}t^9)Y_5(zq^{-11}t^{11})^{-1}Y_7(zq^{-10}t^{10})^{-1} :$
Monomial 37 : $1\cdot : Y_1(zq^{-11}t^{11})Y_2(zq^{-12}t^{12})^{-1}Y_6(zq^{-8}t^8) :$
Monomial 38 : $1\cdot : Y_2(zq^{-10}t^{10})Y_3(zq^{-11}t^{11})^{-1}Y_4(zq^{-10}t^{10})Y_5(zq^{-11}t^{11})^{-1} :$
Monomial 39 : $1\cdot : Y_1(zq^{-11}t^{11})Y_2(zq^{-12}t^{12})^{-1}Y_5(zq^{-9}t^9)Y_6(zq^{-10}t^{10})^{-1} :$
Monomial 40 : $1\cdot : Y_1(zq^{-13}t^{13})^{-1}Y_6(zq^{-8}t^8) :$
Monomial 41 : $1\cdot : Y_1(zq^{-11}t^{11})Y_2(zq^{-12}t^{12})^{-1}Y_4(zq^{-10}t^{10})Y_5(zq^{-11}t^{11})^{-1} :$
Monomial 42 : $1\cdot : Y_2(zq^{-10}t^{10})Y_4(zq^{-12}t^{12})^{-1} :$
Monomial 43 : $1\cdot : Y_1(zq^{-13}t^{13})^{-1}Y_5(zq^{-9}t^9)Y_6(zq^{-10}t^{10})^{-1} :$
Monomial 44 : $1\cdot : Y_1(zq^{-11}t^{11})Y_2(zq^{-12}t^{12})^{-1}Y_3(zq^{-11}t^{11})Y_4(zq^{-12}t^{12})^{-1} :$
Monomial 45 : $1\cdot : Y_1(zq^{-13}t^{13})^{-1}Y_4(zq^{-10}t^{10})Y_5(zq^{-11}t^{11})^{-1} :$
Monomial 46 : $1\cdot : Y_1(zq^{-13}t^{13})^{-1}Y_3(zq^{-11}t^{11})Y_4(zq^{-12}t^{12})^{-1} :$
Monomial 47 : $1\cdot : Y_1(zq^{-11}t^{11})Y_3(zq^{-13}t^{13})^{-1}Y_7(zq^{-12}t^{12}) :$
Monomial 48 : $1\cdot : Y_1(zq^{-13}t^{13})^{-1}Y_2(zq^{-12}t^{12})Y_3(zq^{-13}t^{13})^{-1}Y_7(zq^{-12}t^{12}) :$
Monomial 49 : $1\cdot : Y_1(zq^{-11}t^{11})Y_7(zq^{-14}t^{14})^{-1} :$
Monomial 50 : $1\cdot : Y_1(zq^{-13}t^{13})^{-1}Y_2(zq^{-12}t^{12})Y_7(zq^{-14}t^{14})^{-1} :$
Monomial 51 : $1\cdot : Y_2(zq^{-14}t^{14})^{-1}Y_7(zq^{-12}t^{12}) :$
Monomial 52 : $1\cdot : Y_2(zq^{-14}t^{14})^{-1}Y_3(zq^{-13}t^{13})Y_7(zq^{-14}t^{14})^{-1} :$
Monomial 53 : $1\cdot : Y_3(zq^{-15}t^{15})^{-1}Y_4(zq^{-14}t^{14}) :$
Monomial 54 : $1\cdot : Y_4(zq^{-16}t^{16})^{-1}Y_5(zq^{-15}t^{15}) :$
Monomial 55 : $1\cdot : Y_5(zq^{-17}t^{17})^{-1}Y_6(zq^{-16}t^{16}) :$
Monomial 56 : $1\cdot : Y_6(zq^{-18}t^{18})^{-1} :$

A.1.3. *Type* E_8. In type E_8, the algorithm fails for all $1 \leq i \leq 8$.

A.2. **Type F_4.** In type F_4, there exists fundamental fields associated to the first and fourth fundamental modules of $U_q(\widehat{F_4})$. The algorithm fails to compute $T_2(z)$, $T_3(z)$.
There is the list of monomials in $T_1(z)$:

Monomial 1 : $1\cdot : Y_1(z) :$
Monomial 2 : $1\cdot : Y_1(zq^{-2}t^2)^{-1}Y_2(zq^{-1}t^1) :$
Monomial 3 : $1\cdot : Y_2(zq^{-3}t^3)^{-1}Y_3(zq^{-2}t^2) :$
Monomial 4 : $1\cdot : Y_2(zq^{-5}t^3)Y_3(zq^{-6}t^4)^{-1}Y_4(zq^{-4}t^3) :$
Monomial 5 : $1\cdot : Y_1(zq^{-6}t^4)Y_2(zq^{-7}t^5)^{-1}Y_4(zq^{-4}t^3) :$
Monomial 6 : $1\cdot : Y_2(zq^{-5}t^3)Y_4(zq^{-8}t^5)^{-1} :$
Monomial 7 : $1\cdot : Y_1(zq^{-6}t^4)Y_2(zq^{-7}t^5)^{-1}Y_3(zq^{-6}t^4)Y_4(zq^{-8}t^5)^{-1} :$
Monomial 8 : $1\cdot : Y_1(zq^{-8}t^6)^{-1}Y_4(zq^{-4}t^3) :$
Monomial 9 : $1\cdot : Y_1(zq^{-8}t^6)^{-1}Y_3(zq^{-6}t^4)Y_4(zq^{-8}t^5)^{-1} :$
Monomial 10 : $1\cdot : Y_1(zq^{-6}t^4)Y_2(zq^{-9}t^5)Y_3(zq^{-10}t^6)^{-1} :$
Monomial 11 : $1\cdot : Y_1(zq^{-8}t^6)^{-1}Y_2(zq^{-9}t^5)Y_2(zq^{-7}t^5)Y_3(zq^{-10}t^6)^{-1} :$
Monomial 12 : $1\cdot : Y_1(zq^{-10}t^6)Y_1(zq^{-6}t^4)Y_2(zq^{-11}t^7)^{-1} :$
Monomial 13 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-10}t^6)Y_1(zq^{-8}t^6)^{-1}Y_2(zq^{-11}t^7)^{-1}Y_2(zq^{-7}t^5) :$
Monomial 14 : $\frac{(q^3+t)(q^3-t)(q^2+t^2)(q+t)(q-t)}{(q^3+t^2)(q^3-t^2)(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-12}t^8)^{-1}Y_1(zq^{-6}t^4) :$
Monomial 15 : $1\cdot : Y_1(zq^{-10}t^6)Y_2(zq^{-11}t^7)^{-1}Y_2(zq^{-9}t^7)^{-1}Y_3(zq^{-8}t^6) :$
Monomial 16 : $1\cdot : Y_1(zq^{-12}t^8)^{-1}Y_1(zq^{-8}t^6)^{-1}Y_2(zq^{-7}t^5) :$
Monomial 17 : $1\cdot : Y_1(zq^{-12}t^8)^{-1}Y_2(zq^{-9}t^7)^{-1}Y_3(zq^{-8}t^6) :$
Monomial 18 : $1\cdot : Y_1(zq^{-10}t^6)Y_3(zq^{-12}t^8)^{-1}Y_4(zq^{-10}t^7) :$
Monomial 19 : $1\cdot : Y_1(zq^{-12}t^8)^{-1}Y_2(zq^{-11}t^7)Y_3(zq^{-12}t^8)^{-1}Y_4(zq^{-10}t^7) :$
Monomial 20 : $1\cdot : Y_1(zq^{-10}t^6)Y_4(zq^{-14}t^9)^{-1} :$
Monomial 21 : $1\cdot : Y_2(zq^{-13}t^9)^{-1}Y_4(zq^{-10}t^7) :$
Monomial 22 : $1\cdot : Y_1(zq^{-12}t^8)^{-1}Y_2(zq^{-11}t^7)Y_4(zq^{-14}t^9)^{-1} :$
Monomial 23 : $1\cdot : Y_2(zq^{-13}t^9)^{-1}Y_3(zq^{-12}t^8)Y_4(zq^{-14}t^9)^{-1} :$
Monomial 24 : $1\cdot : Y_2(zq^{-15}t^9)Y_3(zq^{-16}t^{10})^{-1} :$
Monomial 25 : $1\cdot : Y_1(zq^{-16}t^{10})Y_2(zq^{-17}t^{11})^{-1} :$
Monomial 26 : $1\cdot : Y_1(zq^{-18}t^{12})^{-1} :$

There is the list of monomials in $T_4(z)$:

Monomial 1 : $1\cdot : Y_4(z) :$
Monomial 2 : $1\cdot : Y_3(zq^{-2}t^1)Y_4(zq^{-4}t^2)^{-1} :$
Monomial 3 : $1\cdot : Y_2(zq^{-5}t^2)Y_2(zq^{-3}t^2)Y_3(zq^{-6}t^3)^{-1} :$
Monomial 4 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-6}t^3)Y_2(zq^{-7}t^4)^{-1}Y_2(zq^{-3}t^2) :$
Monomial 5 : $1\cdot : Y_1(zq^{-6}t^3)Y_1(zq^{-4}t^3)Y_2(zq^{-7}t^4)^{-1}Y_2(zq^{-5}t^4)^{-1}Y_3(zq^{-4}t^3) :$
Monomial 6 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-8}t^5)^{-1}Y_2(zq^{-3}t^2) :$
Monomial 7 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-8}t^5)^{-1}Y_1(zq^{-4}t^3)Y_2(zq^{-5}t^4)^{-1}Y_3(zq^{-4}t^3) :$
Monomial 8 : $1\cdot : Y_1(zq^{-6}t^3)Y_1(zq^{-4}t^3)Y_3(zq^{-8}t^5)^{-1}Y_4(zq^{-6}t^4) :$
Monomial 9 : $1\cdot : Y_1(zq^{-8}t^5)^{-1}Y_1(zq^{-6}t^5)^{-1}Y_3(zq^{-4}t^3) :$

Monomial 10 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-8}t^5)^{-1}Y_1(zq^{-4}t^3)Y_2(zq^{-7}t^4)Y_3(zq^{-8}t^5)^{-1}Y_4(zq^{-6}t^4)$:

Monomial 11 : $1\cdot : Y_1(zq^{-6}t^3)Y_1(zq^{-4}t^3)Y_4(zq^{-10}t^6)^{-1}$:

Monomial 12 : $1\cdot : Y_1(zq^{-8}t^5)^{-1}Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-7}t^4)Y_2(zq^{-5}t^4)Y_3(zq^{-8}t^5)^{-1}Y_4(zq^{-6}t^4)$:

Monomial 13 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-4}t^3)Y_2(zq^{-9}t^6)^{-1}Y_4(zq^{-6}t^4)$:

Monomial 14 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-8}t^5)^{-1}Y_1(zq^{-4}t^3)Y_2(zq^{-7}t^4)Y_4(zq^{-10}t^6)^{-1}$:

Monomial 15 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-9}t^6)^{-1}Y_2(zq^{-5}t^4)Y_4(zq^{-6}t^4)$:

Monomial 16 : $1\cdot : Y_1(zq^{-8}t^5)^{-1}Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-7}t^4)Y_2(zq^{-5}t^4)Y_4(zq^{-10}t^6)^{-1}$:

Monomial 17 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-4}t^3)Y_2(zq^{-9}t^6)^{-1}Y_3(zq^{-8}t^5)Y_4(zq^{-10}t^6)^{-1}$:

Monomial 18 : $1\cdot : Y_2(zq^{-9}t^6)^{-1}Y_2(zq^{-7}t^6)^{-1}Y_3(zq^{-6}t^5)Y_4(zq^{-6}t^4)$:

Monomial 19 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-9}t^6)^{-1}Y_2(zq^{-5}t^4)Y_3(zq^{-8}t^5)Y_4(zq^{-10}t^6)^{-1}$:

Monomial 20 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-4}t^3)Y_2(zq^{-11}t^6)Y_3(zq^{-12}t^7)^{-1}$:

Monomial 21 : $1\cdot : Y_2(zq^{-9}t^6)^{-1}Y_2(zq^{-7}t^6)^{-1}Y_3(zq^{-8}t^5)Y_3(zq^{-6}t^5)Y_4(zq^{-10}t^6)^{-1}$:

Monomial 22 : $1\cdot : Y_3(zq^{-10}t^7)^{-1}Y_4(zq^{-8}t^6)Y_4(zq^{-6}t^4)$:

Monomial 23 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-11}t^6)Y_2(zq^{-5}t^4)Y_3(zq^{-12}t^7)^{-1}$:

Monomial 24 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-12}t^7)Y_1(zq^{-4}t^3)Y_2(zq^{-13}t^8)^{-1}$:

Monomial 25 : $\frac{(qt+1)(qt-1)}{(q+t)(q-t)}\cdot : Y_3(zq^{-10}t^7)^{-1}Y_3(zq^{-8}t^5)Y_4(zq^{-10}t^6)^{-1}Y_4(zq^{-8}t^6)$:

Monomial 26 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)}\cdot : Y_2(zq^{-11}t^6)Y_2(zq^{-7}t^6)^{-1}Y_3(zq^{-12}t^7)^{-1}Y_3(zq^{-6}t^5)$:

Monomial 27 : $-\frac{(q^3+t)(q^3-t)(t^2+q)(t^2-q)}{(q^3+t^2)(q^3-t^2)(q+t)(q-t)}\cdot : Y_4(zq^{-12}t^8)^{-1}Y_4(zq^{-6}t^4)$:

Monomial 28 : $\frac{(q^4+t)(q^4-t)(q^3+t^2)(q^3-t^2)(q^2+1)(q+t)(q-t)}{(q^4+t^2)(q^3+t)(q^3-t)(q^2+t)^2(q^2-t)^2}\cdot : Y_1(zq^{-12}t^7)Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-13}t^8)^{-1}Y_2(zq^{-5}t^4)$:

Monomial 29 : $\frac{(q^5+t^2)(q^5-t^2)(q^4+t^3)(q^4-t^3)(q^2+1)(q+t)(q-t)}{(q^5+t^3)(q^5-t^3)(q^4+t^2)(q^2+t)^2(q^2-t)^2}\cdot : Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-4}t^3)$:

Monomial 30 : $1\cdot : Y_2(zq^{-11}t^6)Y_2(zq^{-9}t^6)Y_3(zq^{-12}t^7)^{-1}Y_3(zq^{-10}t^7)^{-1}Y_4(zq^{-8}t^6)$:

Monomial 31 : $1\cdot : Y_3(zq^{-8}t^5)Y_4(zq^{-12}t^8)^{-1}Y_4(zq^{-10}t^6)^{-1}$:

Monomial 32 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-12}t^7)Y_2(zq^{-13}t^8)^{-1}Y_2(zq^{-7}t^6)^{-1}Y_3(zq^{-6}t^5)$:

Monomial 33 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-6}t^5)^{-1}Y_2(zq^{-5}t^4)$:

Monomial 34 : $1\cdot : Y_2(zq^{-11}t^6)Y_2(zq^{-9}t^6)Y_3(zq^{-12}t^7)^{-1}Y_4(zq^{-12}t^8)^{-1}$:

Monomial 35 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-12}t^7)Y_2(zq^{-13}t^8)^{-1}Y_2(zq^{-9}t^6)Y_3(zq^{-10}t^7)^{-1}Y_4(zq^{-8}t^6)$:

Monomial 36 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-14}t^9)^{-1}Y_2(zq^{-7}t^6)^{-1}Y_3(zq^{-6}t^5)$:

Monomial 37 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-12}t^7)Y_2(zq^{-13}t^8)^{-1}Y_2(zq^{-9}t^6)Y_4(zq^{-12}t^8)^{-1}$:

Monomial 38 : $1\cdot : Y_1(zq^{-12}t^7)Y_1(zq^{-10}t^7)Y_2(zq^{-13}t^8)^{-1}Y_2(zq^{-11}t^8)^{-1}Y_4(zq^{-8}t^6)$:

Monomial 39 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)}\cdot : Y_1(zq^{-14}t^9)^{-1}Y_2(zq^{-9}t^6)Y_3(zq^{-10}t^7)^{-1}Y_4(zq^{-8}t^6)$:

Monomial 40 : $1\cdot : Y_1(zq^{-12}t^7)Y_1(zq^{-10}t^7)Y_2(zq^{-13}t^8)^{-1}Y_2(zq^{-11}t^8)^{-1}Y_3(zq^{-10}t^7)Y_4(zq^{-12}t^8)^{-1}$:

Monomial 41 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-14}t^9)^{-1}Y_2(zq^{-9}t^6)Y_4(zq^{-12}t^8)^{-1}:$
Monomial 42 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-10}t^7)Y_2(zq^{-11}t^8)^{-1}Y_4(zq^{-8}t^6):$
Monomial 43 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-10}t^7)Y_2(zq^{-11}t^8)^{-1}Y_3(zq^{-10}t^7)Y_4(zq^{-12}t^8)^{-1}:$
Monomial 44 : $1\cdot :Y_1(zq^{-12}t^7)Y_1(zq^{-10}t^7)Y_3(zq^{-14}t^9)^{-1}:$
Monomial 45 : $1\cdot :Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-12}t^9)^{-1}Y_4(zq^{-8}t^6):$
Monomial 46 : $1\cdot :Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-12}t^9)^{-1}Y_3(zq^{-10}t^7)Y_4(zq^{-12}t^8)^{-1}:$
Monomial 47 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-10}t^7)Y_2(zq^{-13}t^8)Y_3(zq^{-14}t^9)^{-1}:$
Monomial 48 : $1\cdot :Y_1(zq^{-14}t^9)^{-1}Y_1(zq^{-12}t^9)^{-1}Y_2(zq^{-13}t^8)Y_2(zq^{-11}t^8)Y_3(zq^{-14}t^9)^{-1}:$
Monomial 49 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-10}t^7)Y_2(zq^{-15}t^{10})^{-1}:$
Monomial 50 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-12}t^9)^{-1}Y_2(zq^{-15}t^{10})^{-1}Y_2(zq^{-11}t^8):$
Monomial 51 : $1\cdot :Y_2(zq^{-15}t^{10})^{-1}Y_2(zq^{-13}t^{10})^{-1}Y_3(zq^{-12}t^9):$
Monomial 52 : $1\cdot :Y_3(zq^{-16}t^{11})^{-1}Y_4(zq^{-14}t^{10}):$
Monomial 53 : $1\cdot :Y_4(zq^{-18}t^{12})^{-1}:$

A.3. **Type G_2.** The algorithm works for $i = 1, 2$. These formulas in type G_2 are also computed by P. Bouwknegt and K. Pilch in [BP98]. Here is the G_2-type Cartan matrix :

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

And

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

There is the list of monomials in $T_1(z)$:

Monomial 1 : $1\cdot :Y_1(z):$
Monomial 2 : $1\cdot :Y_1(zq^{-6}t^2)^{-1}Y_2(zq^{-5}t^1)Y_2(zq^{-3}t^1)Y_2(zq^{-1}t^1):$
Monomial 3 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)} \cdot :Y_2(zq^{-7}t^3)^{-1}Y_2(zq^{-3}t^1)Y_2(zq^{-1}t^1):$
Monomial 4 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)} \cdot :Y_1(zq^{-4}t^2)Y_2(zq^{-7}t^3)^{-1}Y_2(zq^{-5}t^3)^{-1}Y_2(zq^{-1}t^1):$
Monomial 5 : $1\cdot :Y_1(zq^{-4}t^2)Y_1(zq^{-2}t^2)Y_2(zq^{-7}t^3)^{-1}Y_2(zq^{-5}t^3)^{-1}Y_2(zq^{-3}t^3)^{-1}:$
Monomial 6 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)} \cdot :Y_1(zq^{-10}t^4)^{-1}Y_2(zq^{-9}t^3)Y_2(zq^{-1}t^1):$
Monomial 7 : $\frac{(q^4+1)(q^2+1)(q+t)(q-t)}{(q^4+t)(q^4-t)} \cdot :Y_1(zq^{-10}t^4)^{-1}Y_1(zq^{-2}t^2)Y_2(zq^{-9}t^3)Y_2(zq^{-3}t^3)^{-1}:$
Monomial 8 : $\frac{(q^2+1)(qt+1)(qt-1)}{(q^2+t)(q^2-t)} \cdot :Y_1(zq^{-8}t^4)^{-1}Y_1(zq^{-4}t^2):$
Monomial 9 : $\frac{(q^5+t)(q^5-t)(q^4+t^2)(q^2+q+1)(q^2-q+1)(q^2+t)(q^2-t)(q+t)(q-t)}{(q^5+t^2)(q^5-t^2)(q^4+t)(q^4-t)(q^3+t)(q^3-t)} \cdot :Y_2(zq^{-11}t^5)^{-1}Y_2(zq^{-1}t^1):$
Monomial 10 : $1\cdot :Y_1(zq^{-10}t^4)^{-1}Y_1(zq^{-8}t^4)^{-1}Y_2(zq^{-9}t^3)Y_2(zq^{-7}t^3)Y_2(zq^{-5}t^3):$
Monomial 11 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)} \cdot :Y_1(zq^{-2}t^2)Y_2(zq^{-11}t^5)^{-1}Y_2(zq^{-3}t^3)^{-1}:$
Monomial 12 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)} \cdot :Y_1(zq^{-8}t^4)^{-1}Y_2(zq^{-11}t^5)^{-1}Y_2(zq^{-7}t^3)Y_2(zq^{-5}t^3):$

Monomial 13 : $\frac{(q^2+q+1)(q^2-q+1)(q+t)(q-t)}{(q^3+t)(q^3-t)} \cdot : Y_2(zq^{-11}t^5)^{-1}Y_2(zq^{-9}t^5)^{-1}Y_2(zq^{-5}t^3)$:
Monomial 14 : $1\cdot : Y_1(zq^{-6}t^4)Y_2(zq^{-11}t^5)^{-1}Y_2(zq^{-9}t^5)^{-1}Y_2(zq^{-7}t^5)^{-1}$:
Monomial 15 : $1\cdot : Y_1(zq^{-12}t^6)^{-1}$:

There is the list of monomials in $T_2(z)$:

Monomial 1 : $1\cdot : Y_2(z)$:
Monomial 2 : $1\cdot : Y_1(zq^{-1}t^1)Y_2(zq^{-2}t^2)^{-1}$:
Monomial 3 : $1\cdot : Y_1(zq^{-7}t^3)^{-1}Y_2(zq^{-6}t^2)Y_2(zq^{-4}t^2)$:
Monomial 4 : $\frac{(q^2+1)(q+t)(q-t)}{(q^2+t)(q^2-t)} \cdot : Y_2(zq^{-8}t^4)^{-1}Y_2(zq^{-4}t^2)$:
Monomial 5 : $1\cdot : Y_1(zq^{-5}t^3)Y_2(zq^{-8}t^4)^{-1}Y_2(zq^{-6}t^4)^{-1}$:
Monomial 6 : $1\cdot : Y_1(zq^{-11}t^5)^{-1}Y_2(zq^{-10}t^4)$:
Monomial 7 : $1\cdot : Y_2(zq^{-12}t^6)^{-1}$:

B. Proof of Lemma 3 in type B_2, G_2

We prove here Lemma 3 in type B_2, C_2. The proof is quite long, but it closely follows the one for type A_2. The extra length comes simply from the number of cases to check. Indeed, in type B_2, we have to study the nodes associated with both the long and short roots, as well as handle possible cancellations in q and t, while in type A_2 the field is symmetric in q^{-1} and t.

B.1. **In type B_2:** We have $r_1 = 1$, $r_2 = 2$,

$$A_1(z) =: Y_1(zq^{-1}t)Y_1(zqt^{-1})Y_2(z)^{-1} :$$

$$A_2(z) =: Y_2(zq^{-2}t)Y_2(zq^2t^{-1})Y_1(zq^{-1})^{-1}Y_1(zq)^{-1} :$$

B.1.1. *In the case $(i,j) = (1,2)$:*

Proof. We prove it by induction on the height of the monomials. We assume the Lie subalgebra generated by the nodes i and j is of type B_2, and $(i,j) = (1,2)$.
Height 0 : The only monomial with height 0 is the dominant generic monomial from which we start our algorithm. Hence the property at height 0.

Height $h+1$: We assume the property is true for all the monomials with heights $h' \leq h \in \mathbb{N}$. Let us prove it for the monomials with height $h+1$. Let X be a monomial with height $h+1$. The monomial appears in the algorithm, so it has to come from a monomial X' with height h. Let $A_j(zbq^{r_j}t^{-1})^{-1}$ be the transformation involved. It implies that $d_{j,b}(X) = -1$ and $d_{j,bq^{2r_j}t^{-2}}(X') = 1$. Let $i \in I$, $a \in K$ such that $d_{i,a}(X) = -1$ and $d_{i,aq^{2r_i}}(X), d_{i,at^{-2}}(X) \neq -1$. If $(i,a) = (j,b)$ then we have the result. Else, we have

$$X = X'A_j(zbq^{r_j}t^{-1})^{-1}.$$

Hence,

$$\begin{cases} d_{j,bq^{2r_j}t^{-2}}(X') = 1 \\ d_{j,b}(X) = -1 \\ d_{j,c}(X) = d_{j,c}(X') \text{ for all } c \notin \{b, bq^{2r_j}t^{-2}\} \\ d_{k,c}(X) \geq d_{k,c}(X') \text{ for all } k \neq j,\ c \in K \end{cases}$$

We know that $d_{i,a}(X) = -1$, and $(i, a) \neq (j, b)$. It implies $d_{i,a}(X') \leq -1$. The monomials are all generic, thus

$$d_{i,a}(X') = -1$$

Then by the induction assumption, one of the two following assertions is true :

(a) $d_{i,aq^{2r_i}}(X') = -1$ or $d_{i,at^{-2}}(X') = -1$.
(b) There exists a monomial X'' appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X'.$$

a) Firstly we assume (a) is true and X' does contain $Y_i(zaq^{2r_i})^{-1}$ (i.e $d_{i,aq^{2r_i}}(X') = -1$). Let us prove that this case is absurd, then we will consider the case where X' does contain $Y_i(zat^2)^{-1}$ (i.e $d_{i,at^2}(X') = -1$) By assumption, $d_{i,aq^{2r_i}}(X) = 0$. It implies that the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ simplifies $Y_i(zaq^{2r_i})^{-1}$ and in particular $i \neq j$. Therefore, the expressions depend on the type of the Dynkin diagram generated by the nodes i and j.

Let us do it in type B_2, $(i, j) = (1, 2)$.

$$A_j(zbq^{r_j}t^{-1})^{-1} = A_j(zbq^2t^{-1})^{-1} = Y_j(zbq^4t^{-2})^{-1}Y_j(zb)^{-1}Y_i(zbq^3t^{-1})Y_i(zbqt^{-1})$$

Then

$$bq^3t^{-1} = aq^2 \qquad \text{or} \qquad bqt^{-1} = aq^2$$

In the first case, the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ also simplifies $Y_i(zbqt^{-1})^{-1} = Y_i(za)^{-1}$. This is impossible. Thus, we are in the second case and as in type A_2 we get :

$$b = aqt.$$

Let $k > 0$ the maximal integer such that $Y_i(zaq^{2s})^{-1}$ appears in X' for all $0 \leq s \leq k$. We want to prove that the graph representation of the algorithm contains the following subgraph :

$$\begin{array}{cl}
X_{k+1} & \\
\downarrow & A_i(zaqt^{-1})^{-1} \\
\vdots & \\
\downarrow & A_i(zaq^{2k+1}t^{-1})^{-1} \\
X' & \\
\downarrow & A_j(zaq^3)^{-1} \\
X &
\end{array}$$

By definition, $Y_i(zaq^{2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^{2k}t^{-2})^{-1}$ appears in X', then $Y_i(zaq^{2(k-1)})^{-1}Y_i(zaq^{2k}t^{-2})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zaq^{2k})^{-1}Y_i(zaq^{2(k-1)}t^{-2})^{-1}$ appears in X'. Thus, X' contains $Y_i(zaq^{2(k-2)})^{-1}Y_i(zaq^{2(k-1)}t^{-2})^{-1}$. We can iterate this reasoning until we get that $Y_i(zat^{-2})^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^{r_j}t^{-1})^{-1} =$

$A_j(zaq^3)^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1 A_i(zaq^{2k+1}t^{-1})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_s A_i(zaq^{2k-2s+3}t^{-1})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zaq^{2u})^{-1}$ appears in X_s for all $0 \leq u \leq k+1-s$ and $Y_i(zaq^{2(u-1)}t^{-2})^{-1}$ does not appear in X_s nor $Y_i(zaq^{2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1} A_i(zaqt^{-1})^{-1} = X'.$$

But $d_{j,aqt^{-1}}(A_i(zaqt^{-1})^{-1}) = 1$. Thus,

$$d_{j,aqt^{-1}}(X_{k+1}) = d_{j,aqt^{-1}}(X_k) - 1 = d_{j,aqt^{-1}}(X') - 1 < 0$$

because by regularity,

$$d_{j,aqt^{-1}}(X') \in \{0, 1\},$$

and if $d_{j,aqt^{-1}}(X') = 1$ then the transformation $A_j(zbq^2t^{-1})^{-1} = A_j(zaq^3)^{-1}$ would not have been admissible from X'. Thus $d_{j,aqt^{-1}}(X') = 0$ and X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$. Moreover,

$$X_{k+1} A_i(zaqt^{-1})^{-1} A_i(zaq^3t^{-1})^{-1} \cdots A_i(zaq^{2k+1}t^{-1})^{-1} = X'. \tag{33}$$

We recall that $X' A_j(zaq^3)^{-1} = X$. By definition of the algorithm, it implies $Y_j(zaq^5t^{-1})$ is admissible in X'. In particular, it implies that $d_{j,aq^5t^{-1}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (33) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aq^5t^{-1}}(X') \leq d_{j,aq^5t^{-1}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aq^5t^{-1}}(X_{k+1}) \geq 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$ and does not contain $Y_j(zaq^5t^{-1})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1} \dots Y_j(zaqt^{-2s-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^5t^{-2u-1})^{-1}$, then it contains the monomial $Y_j(zaqt^{-2u+1})^{-1} Y_j(zaq^5t^{-2u-1})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^5t^{-2u+1})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaq^5t^{-1})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2} A_j(zaq^3t^{-2(s-u)-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{gathered} X_{k+s+2} A_j(zaq^3t^{-2})^{-1} \cdots A_j(zaq^3t^{-2s-2})^{-1} \times \\ A_i(zaqt^{-1})^{-1} \cdots A_i(zaq^{2k+1}t^{-1})^{-1} A_j(zaq^3)^{-1} = X. \end{gathered} \tag{34}$$

We get the following path in the graph representation of the algorithm:
Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^6t^{-2}) \dots Y_i(zaq^{2k+2}t^{-2}) Y_j(zaq^5t^{-3}) \dots Y_j(zaq^5t^{-2s-3}). \tag{35}$$

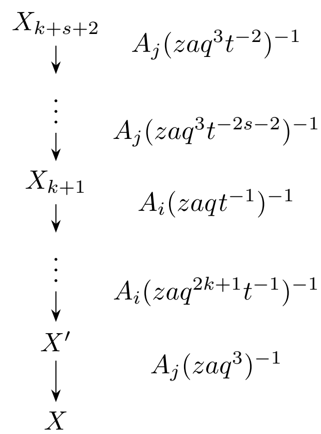
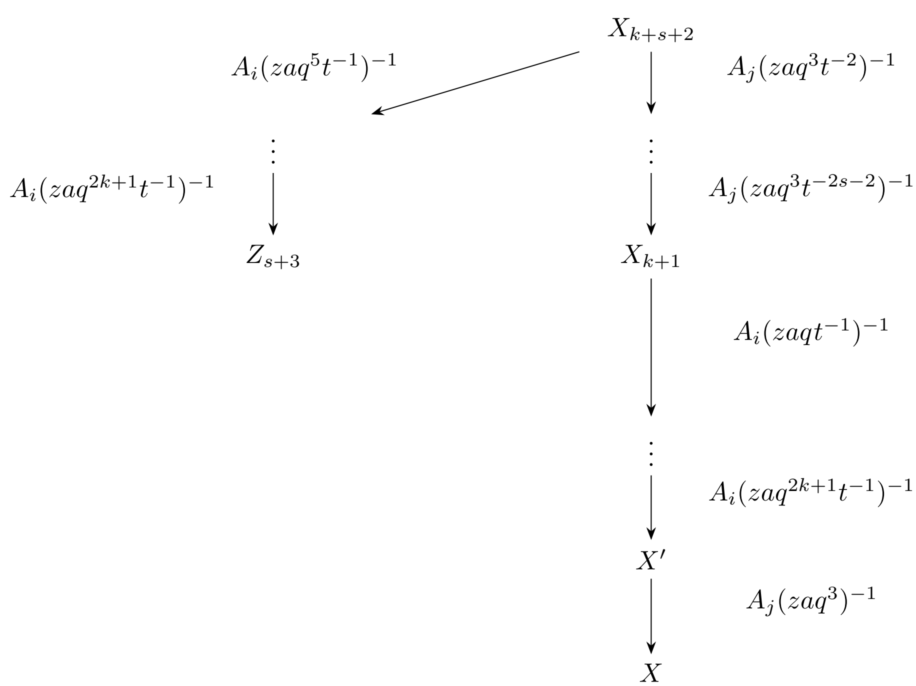


FIGURE 2. One path from X_{k+s+2} to X

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaqt^{-1})^{-1}$ in order to conclude.
We want to obtain the following subgraph :



Firstly, let us prove that $k \geq 2$. The transformation $X' \to X$ gives that $Y_j(zaq^5t^{-1})$ is admissible in X'. Thus, $d_{j,aq^5t^{-1}}(X') = 1$. By contradiction, if $k = 1$ then from the graph below we deduce that

$$d_{j,aq^5t^{-1}}(X') = d_{j,aq^5t^{-1}}(X_{k+s+2}) = 1.$$

Hence, $Y_j(zaq^5t^{-3})$ is not admissible in X_{k+s+2}, which is absurd. Hence, $k \geq 2$.

To apply the transformation $A_i(zaq^5t^{-1})^{-1}$, we need to check that $Y_i(zaq^6t^{-2})$ is admissible in X_{k+s+2}. We know that X_{k+s+2} does not contain $Y_i(zaq^4t^{-2})$ as it would imply that X_{k+s+1} contains $Y_i(zaq^4t^{-2})^2$ which is absurd as all monomials are generic by assumption. Moreover, if X_{k+s+2} contains $Y_i(zaq^6)$ then the equation (49) implies :

$$d_{i,aq^6}(X_{k+s+2}) = d_{i,aq^6}(X_{k-1}) = 1.$$

But the graph gives that $Y_i(zaq^6t^{-2})$ is admissible in X_{k-1}. It is absurd. Thus, $Y_i(zaq^6t^{-2})$ is admissible in X_{k+s+2} and the algorithm apply the transformation $A_i(zaq^5t^{-1})^{-1}$ to X_{k+s+2}, giving a monomial Z_{k+s+1} such that :

$$X_{k+s+2}A_i(zaq^5t^{-1})^{-1} = Z_{k+s+1}. \tag{36}$$

We assume there exists $2 \leq m \leq k-1$ such that the algorithm gives Z_{u+s+1} for all $m < u \leq k$ such that :

$$\forall m < u \leq k, \quad Z_{u+s+2}A_i(zaq^{2(k-u+2)+1}t^{-1})^{-1} = Z_{u+s+1},$$

setting $Z_{k+s+2} = X_{k+s+2}$. We want to prove that the algorithm gives Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{2(k-m+2)+1}t^{-1})^{-1} = Z_{m+s+1}.$$

We know that

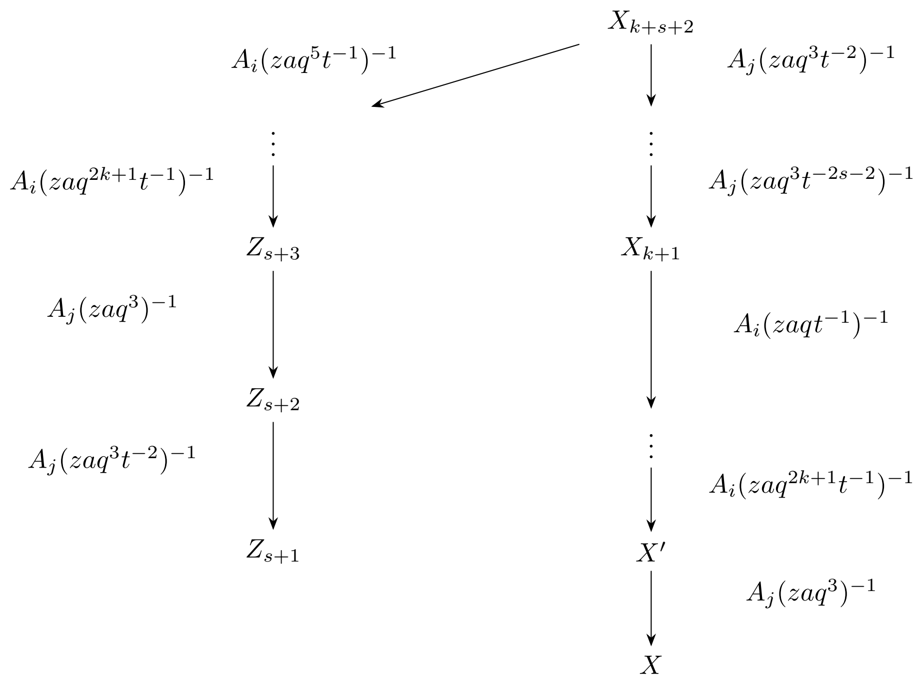
$$Z_{m+s+2} = X_{k+s+2}A_i(zaq^5t^{-1})^{-1} \cdots A_i(zaq^{2(k-m+1)+1}t^{-1})^{-1}. \tag{37}$$

Then, Z_{m+s+2} contains $Y_i(zaq^{2(k-m+3)}t^{-2}) \dots Y_i(zaq^{2k+2}t^{-2})$.
Let us check that $Y_i(zaq^{2(k-m+3)}t^{-2})$ is admissible in Z_{m+s+2}.
It is clear that Z_{m+s+2} does not contain $Y_i(zaq^{2(k-m+2)}t^{-2})$ as it has been removed at the previous step. Moreover, if Z_{m+s+2} contains $Y_i(zaq^{2(k-m+3)})$. then by (37), X_{k+s+2} contains $Y_i(zaq^{2(k-m+3)})$. Thus, according to the graph, X_{m-1} contains $Y_i(zaq^{2(k-m+3)})$. However, we read in the same figure that $Y_i(zaq^{2(k-m+3)}t^{-2})$ is admissible in X_{m-1}. It is absurd. Hence, $Y_i(zaq^{2(k-m+3)}t^{-2})$ is admissible in Z_{m+s+2} and the algorithm gives a monomial Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{2(k-m+2)+1}t^{-1})^{-1} = Z_{m+s+1}.$$

By induction, we get monomials $Z_{s+3}, \dots, Z_{k+s+1}$ verifiying the equation (37).

Now we want to complete our graph to the following :



Furthermore, by (36), we get :

$$d_{j,aq^5t^{-1}}(Z_{s+3}) = d_{j,aq^5t^{-1}}(Z_{k+s+1}) = d_{j,aq^5t^{-1}}(X_{k+s+2}) + 1.$$

But we know by admissibility of $Y_j(zaq^5t^{-3})$ and regularity of X_{k+s+2} that :

$$d_{j,aq^5t^{-1}}(X_{k+s+2}) = 0.$$

Hence, $d_{j,aq^5t^{-1}}(Z_{s+3}) = 1$.
Thus, the monomial Z_{s+3} contains $Y_j(zaq^5t^{-1})\dots Y_j(zaq^5t^{-2s-3})$. We want to check that $Y_j(zaq^5t^{-1})$ is admissible in Z_{s+3}.
If Z_{s+3} contains $Y_j(zaqt^{-1})$ then it contains $Y_j(zaq^5t^{-3})Y_j(zaqt^{-1})$ (this is of the form $Y_j(zc)Y_j(zcq^{-2r_j}t^2)$). By regularity, it also contains $Y_j(zaqt^{-3})$. By (37), X_{k+s+2} contains $Y_j(zaqt^{-3})$, and $Y_j(zaq^5t^{-3})$ is not admissible in X_{k+s+2}. It is absurd.
If Z_{s+3} contains $Y_j(zaq^5t)$ then by (37), X_{k+s+2} contains $Y_j(zaq^5t)$. According to Figure 3, this implies that X' contains $Y_j(zaq^5t)$. But $Y_j(zaq^5t^{-1})$ is admissible in X'. It is absurd.
Hence, $Y_j(zaq^5t^{-1})$ is admissible in Z_{s+3} and the algorithm gives a monomial Z_{s+2} such that :

$$Z_{s+3}A_j(zaq^3)^{-1} = Z_{s+2}. \tag{38}$$

Now we want to construct Z_{s+1} such that

$$Z_{s+2}A_j(zaq^3t^{-2})^{-1} = Z_{s+1}.$$

Let us prove that $Y_j(zaq^5t^{-3})$ is admissible in Z_{s+2}.
By admissibility,

$$d_{j,aq^5t^{-3}}(Z_{s+2}) = d_{j,aq^5t^{-3}}(X_{k+s+2}) = 1.$$

Moreover,

$$d_{j,aqt^{-3}}(Z_{s+2}) = d_{j,aqt^{-3}}(X_{k+s+2}) = 0,$$

and

$$d_{j,aq^5t^{-1}}(Z_{s+2}) = d_{j,aq^5t^{-1}}(Z_{s+3}) - 1 = 1 - 1 = 0.$$

Hence the admissibility and the monomial Z_{s+1} appears in the algorithm.

Finally, let us prove that Z_{s+1} is not regular.
By regularity and because $Y_i(zaq^2t^{-2})$ is admissible in X_{k+1}, we have $d_{i,aq^2}(X_{k+1}) = 0$. Thus,

$$\begin{aligned} d_{i,aq^2}(Z_{s+1}) &= d_{i,aq^2}(Z_{s+2}), \\ &= d_{i,aq^2}(Z_{s+3}) + 1, \\ &= d_{i,aq^2}(X_{k+s+2}) + 1, \\ &= d_{i,aq^2}(X_{k+1}) + 1, \\ d_{i,aq^2}(Z_{s+1}) &= 1. \end{aligned}$$

Moreover, by admissibility, $d_{i,aq^4t^{-2}}(X_k) = 1$. Thus,

$$\begin{aligned} d_{i,aq^4t^{-2}}(Z_{s+1}) &= d_{i,aq^4t^{-2}}(Z_{s+2}) + 1, \\ &= d_{i,aq^4t^{-2}}(X_{k+s+2}) + 1, \\ &= d_{i,aq^4t^{-2}}(X_{k+s+1}) - 1 + 1, \\ &= d_{i,aq^4t^{-2}}(X_k), \\ d_{i,aq^4t^{-2}}(Z_{s+1}) &= 1. \end{aligned}$$

However, by construction of the algorithm, the transformation $X_{k+s+2} \to Z_{k+s+1}$ gives that $d_{i,aq^4}(Z_{k+s+1}) = -1$. Thus the graph implies

$$\begin{aligned} d_{i,aq^4}(Z_{s+1}) &= d_{i,aq^4}(Z_{s+2}), \\ &= d_{i,aq^4}(Z_{s+3}) + 1, \\ &= d_{i,aq^4}(Z_{k+s+1}) + 1, \\ d_{i,aq^4}(Z_{s+1}) &= 0. \end{aligned}$$

Hence setting $c = q^4t^{-2}$, Z_{s+1} contains $Y_i(zc)Y_i(zcq^{-2r_i}t^2)$ but does not contain $Y_i(zct^2)$. It is absurd as assumed that the algorithm does not fail.

Now, we assume $A_j(zbq^{r_j}t^{-1})$ simplifies $Y_i(zat^2)$.

Now, we assume $d_{i,at^{-2}}(X') = -1$. We know that $d_{i,at^{-2}}(X) = 0$, then it implies

$$d_{i,at^{-2}}(A_j(zbq^2t^{-1})^{-1}) = 1.$$

Hence,

$$bq^3t^{-1} = at^{-2} \qquad \text{or} \qquad bqt^{-1} = at^{-2}.$$

We assume $bqt^{-1} = at^{-2}$. Firstly, it is clear that $d_{i,a}(X') = -1$ and $d_{i,aq^2}(X') = 0$. Then by regularity, $d_{i,aq^2t^{-2}}(X') \neq -1$. Moreover,

$$d_{i,aq^2t^{-2}}(A_j(zbq^2t^{-1})^{-1}) = 1.$$

Then, $d_{i,aq^2t^{-2}}(X') = d_{i,aq^2t^{-2}}(X) - 1$. Thus, $d_{i,aq^2t^{-2}}(X) = 1$ (all degrees are in $\{-1, 0, 1\}$ by genericity). Hence, X contains $Y_i(zaq^2t^{-2})Y_i(za)^{-1}$. Let us prove that $Y_i(zaq^2t^{-2})$ is admissible in X. If $d_{i,aq^2}(X) = 1$ (resp. $d_{i,at^{-2}}(X) = 1$) then X is not regular as it contains a term of the form $Y_i(zc)Y_i(zcq^{-2r_i})^{-1}$ (resp. $Y_i(zc)Y_i(zct^2)^{-1}$). Hence the admissibility of $Y_i(zaq^2t^{-2})$in X. So, the algorithm implies a transformation to obtain the monomial

$$X^* = XA_izaqt^{-1},$$

and $d_{i,a}(X^*) = -2$ which contradicts the genericity.

Hence, $b = aq^{-3}t^{-1}$. As in the previous case, we want to apply the induction hypothesis and get the following graph for a k defined above :

$$\begin{array}{cl}
X_{k+1} & \\
\downarrow & A_i(zaqt^{-1})^{-1} \\
\vdots & \\
\downarrow & A_i(zaqt^{-2k-1})^{-1} \\
X' & \\
\downarrow & A_j(zaq^{-1}t^{-2})^{-1} \\
X &
\end{array}$$

Let $k > 0$ the maximal integer such that $Y_i(zat^{-2s})^{-1}$ appears in X' for all $0 \leq s \leq k$.
By definition, $Y_i(zat^{-2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^2t^{-2k})^{-1}$ appears in X', then $Y_i(zat^{-2(k-1)})^{-1}Y_i(zaq^2t^{-2k})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zat^{-2k})^{-1}Y_i(zaq^2t^{-2(k-1)})^{-1}$ appears in X'.
Thus, X' contains $Y_i(zat^{-2(k-2)})^{-1}Y_i(zaq^2t^{-2(k-1)})^{-1}$. We can iterate this reasoning until we get that $Y_i(zaq^2)^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^2t^{-1})^{-1} = A_j(zaq^{-1}t^{-2})^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1A_i(zaqt^{-(2k+1)})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_sA_i(zaqt^{-(2k-2s+3)})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zat^{-2u})^{-1}$ appears in X_s for all $0 \leq u \leq k-s$ and $Y_i(zaq^2t^{-2(u-1)})^{-1}$ does not appear in X_s nor $Y_i(zat^{-2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1}A_i(zaqt^{-1})^{-1} = X_k.$$

Thus,

$$X_{k+1}A_i(zaqt^{-1})^{-1}A_i(zaqt^{-3})^{-1}\cdots A_i(zaqt^{-(2k+1)})^{-1} = X'. \tag{39}$$

By definition of the algorithm, the arrow $X' \to X$ implies that $Y_j(zaqt^{-3})$ is admissible in X'. In particular, it implies that $d_{j,aqt^{-3}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (39) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aqt^{-3}}(X') = d_{j,aqt^{-3}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aqt^{-3}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$ and does not contain $Y_j(zaqt^{-3})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1} \dots Y_j(zaq^{4s+1}t^{-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^{4u+1}t^{-3})^{-1}$, then it contains the monomial $Y_j(zaq^{4(u-1)+1}t^{-1})^{-1}Y_j(zaq^{4u+1}t^{-3})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^{4(u-1)+1}t^{-3})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaqt^{-3})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^{4(s-u)+3}t^{-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{aligned} &X_{k+s+2}A_j(zaq^3t^{-2})^{-1} \cdots A_j(zaq^{4s+3}t^{-2})^{-1} \times \\ &A_i(zaqt^{-1})^{-1} \cdots A_i(zaqt^{-2k-1})^{-1} A_j(zaq^{-1}t^{-2})^{-1} = X. \end{aligned} \tag{40}$$

We get the following path in the graph representation of the algorithm:

$$\begin{array}{cl} X_{k+s+2} & \\ \downarrow & A_j(zaq^3t^{-2})^{-1} \\ \vdots & \\ \downarrow & A_j(zaq^{4s+3}t^{-2})^{-1} \\ X_{k+1} & \\ \downarrow & A_i(zaqt^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaqt^{-2k-1})^{-1} \\ X' & \\ \downarrow & A_j(zaq^{-1}t^{-2})^{-1} \\ X & \end{array}$$

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^2t^{-4}) \dots Y_i(zaq^2t^{-2k-2}) Y_j(zaq^5t^{-3}) \dots Y_j(zaq^{4s+5}t^{-3}). \tag{41}$$

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaqt^{-1})^{-1}$ in order to conclude.
Firstly,

$$d_{i,aq^2t^{-4}}(X_{k+s+2}) = d_{i,aq^2t^{-4}}(X_k) = 1,$$

because as shown in the graph below, $Y_i(zaq^2t^{-4})$ is admissible in X_k. Moreover, $d_{i,at^4}(X_{k+s+2}) = d_{i,at^{-4}}(X_k) = 0$ for the same argument of admissibility. Additionally, if $d_{i,aq^2t^{-2}}(X_{k+s+2}) = 1$ then by definition of $A_j(zaq^3t^{-2})^{-1}$ it implies $d_{i,aq^2t^{-2}}(X_{k+s+1}) = 2$ which is impossible by the genericity hypothesis.
Hence, $Y_i(zaq^2t^{-4})$ is admissible in X_{k+s+2}.
We construct

$$Z_{k+s+1} := X_{k+s+2}A_i(zaqt^{-3})^{-1}.$$

We assume there exists $1 \leq m \leq k$ such that we constructed Z_{s+u+1}, for $m < u \leq k$ such that

$$Z_{u+s+1} = Z_{u+s+2}A_i(zaqt^{-2(k-u)-3})^{-1}.$$

Let us construct Z_{s+m+1}.
Firstly, according to the graph,

$$d_{i,aq^2t^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,aq^2t^{-2(k-m)-4}}(X_m) = 1.$$

Moreover,

$$d_{i,aq^2t^{-2(k-m)-2}}(Z_{s+m+2}) = d_{i,aq^2t^{-2(k-m)-2}}(Z_{s+m+3}) - 1 = d_{i,aq^2t^{-2(k-m)-2}}(X_{k+s+2}) - 1 = 0,$$

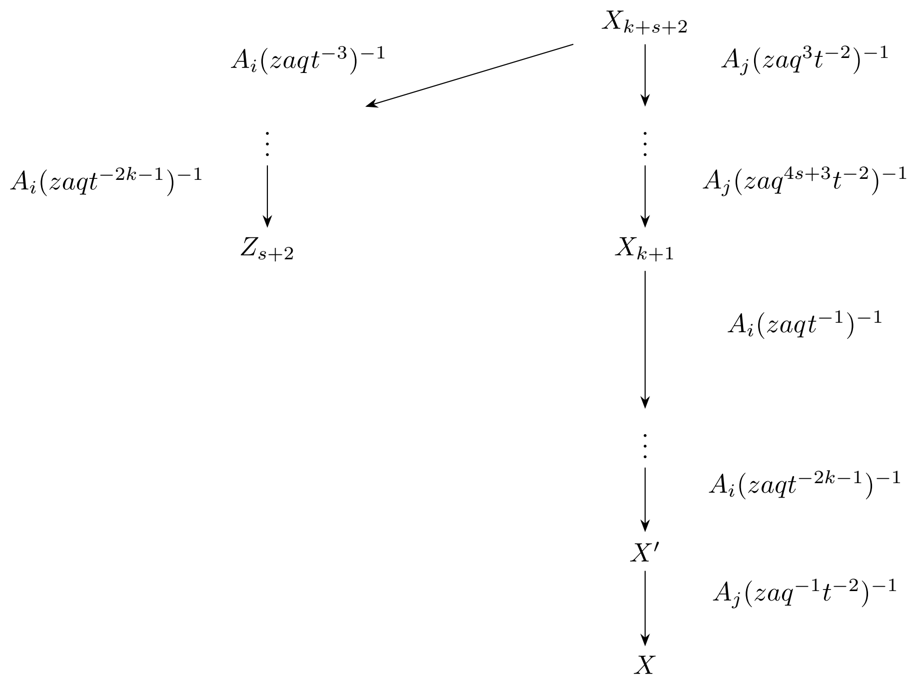
because of the transformation $Z_{s+m+3} \to Z_{s+m+2}$.
Additionally, $Y_i(zaq^2t^{-2(k-m)-4})$ is admissible in X_m. Thus,

$$d_{i,at^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,at^{-2(k-m)-4}}(X_{k+s+2}) = d_{i,at^{-2(k-m)-4}}(X_m) = 0.$$

Hence the admissibility of $Y_i(zaq^2t^{-2(k-m)-4})$ in Z_{s+m+2} and the existence of the following transformation given by the algorithm :

$$Z_{m+s+1} = Z_{m+s+2}A_i(zaqt^{-2(k-u)-3})^{-1}.$$

We obtain the following graph :



Again, we want to construct a path from Z_{s+2} to X in the algorithm graph.
Firstly,

$$\begin{aligned} d_{j,aqt^{-3}}(Z_{s+2}) &= d_{j,aqt^{-3}}(X_{k+s+2}) + 1, \\ &= d_{j,aqt^{-3}}(X_k) + 1, \\ &= d_{j,aqt^{-3}}(X_{k-1}) - 1 + 1, \\ &= d_{j,aqt^{-3}}(X') = 1. \end{aligned}$$

Because $Y_j(zaqt^{-3})$ is admissible in X'.
Moreover,

$$d_{j,aq^{-3}t^{-3}}(Z_{s+2}) = d_{j,aq^{-3}t^{-3}}(X') = 0,$$

as $Y_j(zaqt^{-3})$ is admissible in X'. Additionally, we deduce from the graph the following computation

$$\begin{aligned} d_{j,aqt^{-1}}(Z_{s+2}) &= d_{j,aqt^{-1}}(X_{k+s+2}), \\ &= d_{j,aqt^{-1}}(X_{k+s+1}) - 1, \\ &= d_{j,aqt^{-1}}(X_k) + 1 - 1, \\ &= d_{j,aqt^{-1}}(X') = 0. \end{aligned}$$

Hence, $Y_j(zaqt^{-3})$ is admissible in Z_{s+2}. Thus, the algorithm constructs the following monomial :

$$Z_{s+1} = Z_{s+2}A_j(zaq^{-1}t^{-2})^{-1}.$$

Now, we assume there exists $0 \leq m \leq s$ such that we constructed Z_{u+1}, for $m \leq u \leq s$ such that

$$Z_{u+1} = Z_{u+2} A_j(zaq^{4(s-u)-1}t^{-2})^{-1}.$$

Let us construct Z_m.
Let us prove that $Y_j(zaq^{4(s-m)+5}t^{-3})$ is admissible in Z_{m+1}.
Firstly, according to the graph and by admissibility,

$$d_{j,aq^{4(s-m)+5}t^{-3}}(Z_{m+1}) = d_{j,aq^{4(s-m)+5}t^{-3}}(X_{k+m+2}) = 1.$$

Moreover,

$$d_{j,aq^{4(s-m)+1}t^{-3}}(Z_{m+1}) = d_{j,aq^{4(s-m)+1}t^{-3}}(Z_{m+2}) - 1 = 0,$$

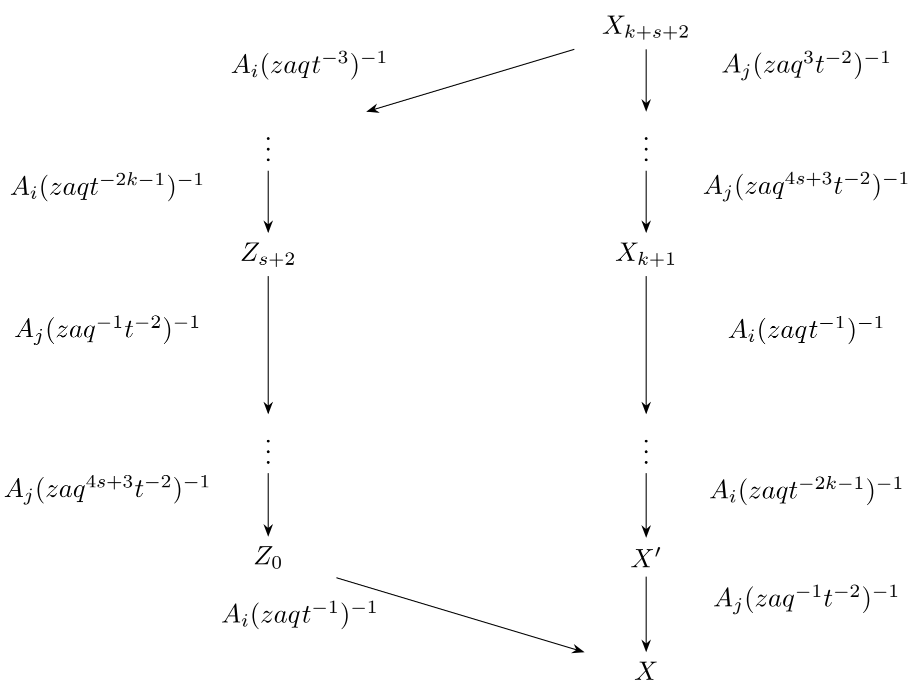
by admissibility and because of the transformation $Z_{m+2} \to Z_{m+1}$.
Additionally, $Y_j(zaq^{4(s-m)+5}t^{-3})$ is admissible in X_{k+m+2}. Thus,

$$d_{j,aq^{4(s-m)+5}t^{-1}}(Z_{m+1}) = d_{j,aq^{4(s-m)+5}t^{-1}}(X_{k+m+2}) = 0.$$

Hence the admissibility of $Y_j(zaq^{4(s-m)+5}t^{-3})$ in Z_{m+1} and the existence of the following transformation given by the algorithm :

$$Z_m = Z_{m+1} A_j(zaq^{4(s-m)+3}t^{-2})^{-1}.$$

To conclude, we have to prove that $Y_i(zaq^2t^{-2})$ is admissible in Z_0.
We have the equality $Z_0 A_i(zqt^{-1})^{-1} = X$. Then, by identification of the degrees it is clear.
Hence the following paths of transformations in the algorithm :



b) We assume there exists a monomial X'' also appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X',$$

then X'' does contain $Y_i(zaq^{2r_i}t^{-2})$.

Here there are two cases to consider :

- $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_j(zaq^{2r_j}t^{-2})$ in X' comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$.
- X'' contains $Y_j(zaq^{2r_j}t^{-2})$.

Firstly, we assume $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_i(zaq^{2r_i}t^{-2})$ comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$. Hence, $i \neq j$, and we assume the Dynkin subdiagram with nodes $\{i,j\}$ is of type B_2 with $(i,j) = (1,2)$. Thus, $bq^4t^{-2} = aqt^{-1}$, and $bq^3t^{-1} = a$. Hence,

$$d_{i,a}(A_j(zbq^2t^{-1})^{-1}) = 1.$$

However, $d_{i,a}(X) = -1$ and $X'A_j(zaq^2t^{-1})^{-1} = X$. Thus, $d_{i,a}(X') = -2$ which contradicts the genericity. It is absurd.

Hence, we are in the second case and X'' contains $Y_j(zaq^{2r_j}t^{-2})$.
Let us prove that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i = j$ (so that $r_i = r_j$), then by contradiction we assume that $Y_j(zbq^{2r_j}t^{-2})$ is not admissible in X''. We know that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X' so that the presence of $Y_i(zaq^{2r_i}t^{-2})$ in X'' prevents from doing the transformation $A_j(bq^{r_j}t^{-1})^{-1}$. It implies that $a = bq^{-2r_i}$ or $a = bt^2$. This implies X contains $Y_i(zb)^{-1} = Y_i(zaq^{2r_i})^{-1}$ or $Y_i(zat^{-2})^{-1}$. It is absurd by the initial assumption on X. Thus, $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i \neq j$ then it is clear that the admissibility of $Y_j(zbq^{2r_j}t^(-2))$ in X' implies its admissibility in X''.
Finally, there exists a monomial

$$X_{new} = X''A_j(zaq^{r_j}t^{-1})^{-1}$$

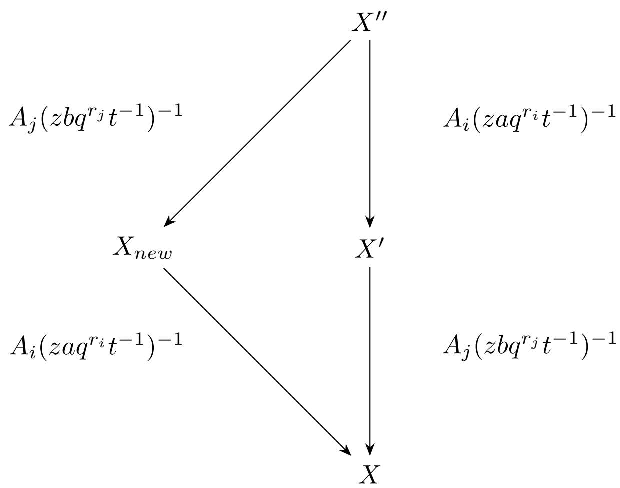
given by the algorithm and X_{new} contains $Y_i(zaq^{2r_i}t^{-2})$. Moreover, if the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$ is not admissible from X_{new} then there exists a factor $Y_i(zat^{-2})$ (resp. $Y_i(zaq^{2r_i})$) blocking this transformation. But this would imply that this factor also appears in X as we have the following equality of monomials :

$$X_{new} = A_i(zaq^{r_i}t^{-1})^{-1}X.$$

We remark that a priori, this equality is not sufficient to get an arrow from X_{new} to X. This equality implies :

$$d_{i,aq^{2r_i}}(X_{new}) = d_{i,aq^{2r_i}}(X),$$

and same for at^{-2}. And hence X does contain $Y_i(zc)$ and $Y_i(zct^2)^{-1}$ with $c = at^{-2}$ (resp. $Y_i(zcq^{-2r_i})^{-1}$ with $c = aq^{2r_i}$) which contradicts the regularity of X. It is impossible. Hence the admissibility of $Y_i(zaq^{2r_i}t^{-2})$ in X_{new}.
Finally we get the following paths :



and hence the result. This concludes the proof in type B_2 for $(i,j) = (1,2)$.

□

B.1.2. *In the case* $(i,j) = (2,1)$ *:*

Proof. We prove it by induction on the height of the monomials. We assume the Lie subalgebra generated by the nodes i and j is of type B_2, and $(i,j) = (2,1)$. Height 0 *:* The only monomial with height 0 is the dominant generic monomial from which we start our algorithm. Hence the property at height 0.

Height $h+1$ *:* We assume the property is true for all the monomials with heights $h' \leq h \in \mathbb{N}$. Let us prove it for the monomials with height $h+1$. Let X be a monomial with height $h+1$. The monomial appears in the algorithm, so it has to come from a monomial X' with height h. Let $A_j(zbq^{r_j}t^{-1})^{-1}$ be the transformation involved. It implies that $d_{j,b}(X) = -1$ and $d_{j,bq^{2r_j}t^{-2}}(X') = 1$. Let $i \in I$, $a \in K$ such that $d_{i,a}(X) = -1$ and $d_{i,aq^{2r_i}}(X), d_{i,at^{-2}}(X) \neq -1$. If $(i,a) = (j,b)$ then we have the result. Else, we have

$$X = X' A_j(zbq^{r_j}t^{-1})^{-1}.$$

Hence,

$$\begin{cases} d_{j,bq^{2r_j}t^{-2}}(X') = 1 \\ d_{j,b}(X) = -1 \\ d_{j,c}(X) = d_{j,c}(X') \text{ for all } c \notin \{b, bq^{2r_j}t^{-2}\} \\ d_{k,c}(X) \geq d_{k,c}(X') \text{ for all } k \neq j,\ c \in K \end{cases}$$

We know that $d_{i,a}(X) = -1$, and $(i,a) \neq (j,b)$. It implies $d_{i,a}(X') \leq -1$. The monomials are all generic, thus

$$d_{i,a}(X') = -1$$

Then by the induction assumption, one of the two following assertions is true :

(a) $d_{i,aq^{2r_i}}(X') = -1$ or $d_{i,at^{-2}}(X') = -1$.

(b) There exists a monomial X'' appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X'.$$

a) Firstly we assume (a) is true and X' does contain $Y_i(zaq^{2r_i})^{-1}$ (i.e $d_{i,aq^{2r_i}}(X') = -1$). By assumption, X does not. It implies that the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ simplifies $Y_i(zaq^{2r_i})^{-1}$ and in particular $i \neq j$. Therefore, the expressions depend on the type of the Dynkin diagram generated by the nodes i and j.

Let us do it in type B_2, $(i,j) = (2,1)$. Thus, $(r_i, r_j) = (2,1)$. Then $bqt^{-1} = aq^4$ and $b = aq^3t$
Let $k > 0$ the maximal integer such that $Y_i(zaq^{4s})^{-1}$ appears in X' for all $0 \leq s \leq k$. We want to prove that the graph representation of the algorithm contains the following subgraph :

$$\begin{array}{cl} X_{k+1} & \\ \downarrow & A_i(zaq^2t^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaq^{4k+2}t^{-1})^{-1} \\ X' & \\ \downarrow & A_j(zaq^4)^{-1} \\ X & \end{array}$$

By definition, $Y_i(zaq^{4(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^{4k}t^{-2})^{-1}$ appears in X', then $Y_i(zaq^{4(k-1)})^{-1}Y_i(zaq^{4k}t^{-2})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zaq^{4k})^{-1}Y_i(zaq^{4(k-1)}t^{-2})^{-1}$ appears in X'.
Thus, X' contains $Y_i(zaq^{4(k-2)})^{-1}Y_i(zaq^{4(k-1)}t^{-2})^{-1}$. We can iterate this reasoning until we get that $Y_i(zat^{-2})^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^{r_j}t^{-1})^{-1} = A_j(zaq^4)^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1A_i(zaq^{4k+2}t^{-1})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_sA_i(zaq^{4k-4s+6}t^{-1})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zaq^{4u})^{-1}$ appears in X_s for all $0 \leq u \leq k-s$ and $Y_i(zaq^{4(u-1)}t^{-2})^{-1}$ does not appear in X_s nor $Y_i(zaq^{4(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1}A_i(zaq^2t^{-1})^{-1} = X_k.$$

Moreover,

$$X_{k+1}A_i(zaq^2t^{-1})^{-1}A_i(zaq^6t^{-1})^{-1} \cdots A_i(zaq^{4k+2}t^{-1})^{-1} = X'. \tag{42}$$

We recall that $X'A_j(zaq^4)^{-1} = X$. By definition of the algorithm, it implies $Y_j(zaq^5t^{-1})$ is admissible in X'. In particular, it implies that $d_{j,aq^5t^{-1}}(X') = 1$ and $d_{j,aq^3t^{-1}}(X') = 0$. But the equation (33) gives

$$d_{j,aq^3t^{-1}}(X') = d_{j,aq^3t^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aq^5t^{-1}}(X') \leq d_{j,aq^5t^{-1}}(X_{k+1}) + 1.$$

Hence, $d_{j,aq^3t^{-1}}(X_{k+1}) = -1$ and $d_{j,aq^5t^{-1}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaq^3t^{-1})^{-1}$ and does not contain $Y_j(zaq^5t^{-1})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaq^3t^{-1})^{-1} \dots Y_j(zaq^3t^{-2s-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^5t^{-2u-1})^{-1}$, then it contains the monomial $Y_j(zaq^3t^{-2u+1})^{-1}Y_j(zaq^5t^{-2u-1})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^5t^{-2u+1})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaq^5t^{-1})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^4t^{-2(s-u)-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$X_{k+s+2}A_j(zaq^4t^{-2})^{-1} \cdots A_j(zaq^4t^{-2s-2})^{-1} \times$$
$$A_i(zaq^2t^{-1})^{-1} \cdots A_i(zaq^{4k+2}t^{-1})^{-1}A_j(zaq^4)^{-1} = X. \tag{43}$$

We get the following path in the graph representation of the algorithm:

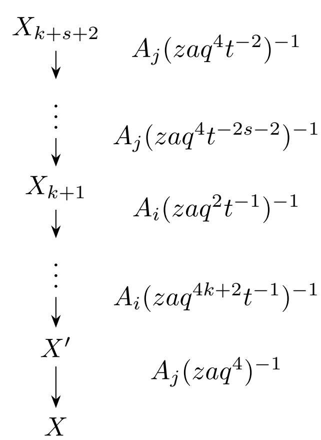
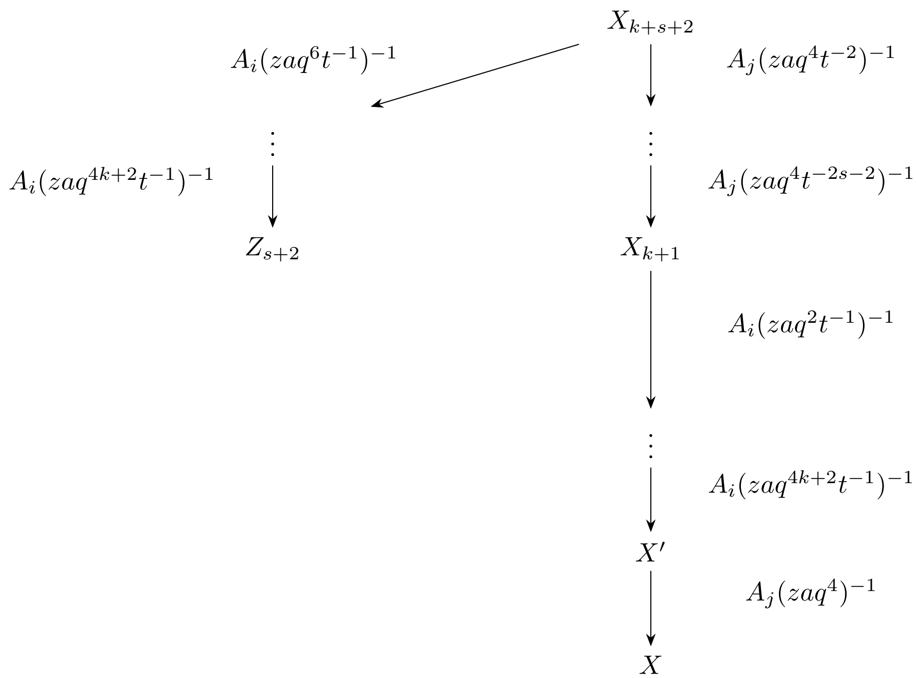


FIGURE 3. One path from X_{k+s+2} to X

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^8t^{-2}) \dots Y_i(zaq^{4k+4}t^{-2})Y_j(zaq^5t^{-3}) \dots Y_j(zaq^5t^{-2s-3}). \tag{44}$$

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaq^2t^{-1})^{-1}$ in order to conclude.
We want to obtain the following subgraph :



To apply the transformation $A_i(zaq^6t^{-1})^{-1}$, we need to check that $Y_i(zaq^8t^{-2})$ is admissible in X_{k+s+2}. We know that X_{k+s+2} does not contain $Y_i(zaq^4t^{-2})$ as it would imply that X_{k+s+1} contains $Y_i(zaq^4t^{-2})^2$ which is absurd as all monomials are generic by assumption. Moreover, if X_{k+s+2} contains $Y_i(zaq^8)$ then the equation (43) implies :

$$d_{i,aq^8}(X_{k+s+2}) = d_{i,aq^8}(X_k) = 1.$$

But the graph gives that $Y_i(zaq^8t^{-2})$ is admissible in X_k. It is absurd. Thus, $Y_i(zaq^8t^{-2})$ is admissible in X_{k+s+2} and the algorithm applies the transformation $A_i(zaq^6t^{-1})^{-1}$ to X_{k+s+2}, giving a monomial Z_{k+s+1} such that :

$$X_{k+s+2}A_i(zaq^6t^{-1})^{-1} = Z_{k+s+1}. \tag{45}$$

We assume there exists $1 \leq m \leq k-1$ such that the algorithm gives Z_{u+s+1} for all $m < u \leq k$ such that :

$$\forall m < u \leq k, \quad Z_{u+s+2}A_i(zaq^{4(k-u+1)+2}t^{-1})^{-1} = Z_{u+s+1},$$

setting $Z_{k+s+2} = X_{k+s+2}$. We want to prove that the algorithm gives Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{4(k-m+1)+2}t^{-1})^{-1} = Z_{m+s+1}.$$

We know that

$$Z_{m+s+2} = X_{k+s+2}A_i(zaq^6t^{-1})^{-1}\cdots A_i(zaq^{4(k-m)+2}t^{-1})^{-1}. \tag{46}$$

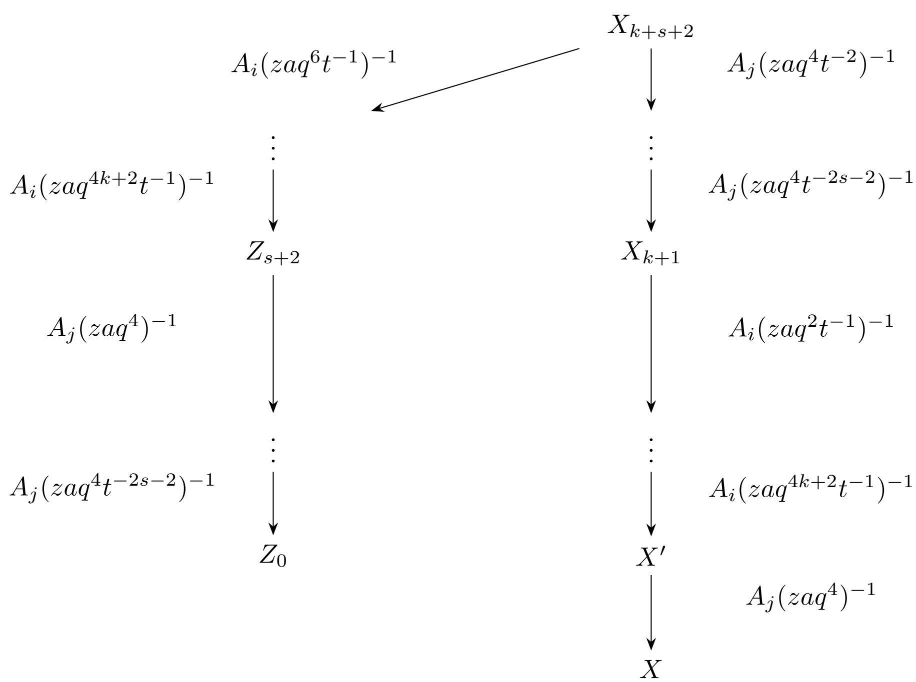
Then, Z_{m+s+2} contains $Y_i(zaq^{4(k-m+2)}t^{-2})\dots Y_i(zaq^{4k+4}t^{-2})$.
Let us check that $Y_i(zaq^{4(k-m+2)}t^{-2})$ is admissible in Z_{m+s+2}.
It is clear that Z_{m+s+2} does not contain $Y_i(zaq^{4(k-m+1)}t^{-2})$ as it has been removed at the previous step.

Moreover, if Z_{m+s+2} contains $Y_i(zaq^{4(k-m+2)})$. then by (46), X_{k+s+2} contains $Y_i(zaq^{4(k-m+2)})$. Thus, according to the graph, X_m contains $Y_i(zaq^{4(k-m+2)})$. However, we read in the same figure that $Y_i(zaq^{4(k-m+2)}t^{-2})$ is admissible in X_m. It is absurd. Hence, $Y_i(zaq^{4(k-m+2)}t^{-2})$ is admissible in Z_{m+s+2} and the algorithm gives a monomial Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{4(k-m+1)+2}t^{-1})^{-1} = Z_{m+s+1}.$$

By induction, we get monomials $Z_{s+2}, \ldots, Z_{k+s+1}$ verifiying the equation (46).

Now we want to complete our graph to the following :



Furthermore, by (45), we get :

$$d_{j,aq^5t^{-1}}(Z_{s+2}) = d_{j,aq^5t^{-1}}(Z_{k+s+1}) = d_{j,aq^5t^{-1}}(X_{k+s+2}) + 1.$$

But we know by admissibility of $Y_j(zaq^5t^{-3})$ and regularity of X_{k+s+2} that :

$$d_{j,aq^5t^{-1}}(X_{k+s+2}) = 0.$$

Hence, $d_{j,aq^5t^{-1}}(Z_{s+2}) = 1$.
Thus, the monomial Z_{s+2} contains $Y_j(zaq^5t^{-1})\ldots Y_j(zaq^5t^{-2s-3})$. We want to check that $Y_j(zaq^5t^{-1})$ is admissible in Z_{s+2}.
If Z_{s+2} contains $Y_j(zaq^3t^{-1})$ then it contains $Y_j(zaq^5t^{-3})Y_j(zaq^3t^{-1})$ (this is of the form $Y_j(zc)Y_j(zcq^{-2r_j}t^2)$). By regularity, it also contains $Y_j(zaq^3t^{-3})$. By (46), X_{k+s+2} contains $Y_j(zaq^3t^{-3})$, and $Y_j(zaq^5t^{-3})$ is not admissible in X_{k+s+2}. It is absurd.
If Z_{s+2} contains $Y_j(zaq^5t)$ then by (46), X_{k+s+2} contains $Y_j(zaq^5t)$. , According to the graph, this implies

that X' contains $Y_j(zaq^5t)$. But $Y_j(zaq^5t^{-1})$ is admissible in X'. It is absurd.
Hence, $Y_j(zaq^5t^{-1})$ is admissible in Z_{s+2} and the algorithm gives a monomial Z_{s+1} such that :

$$Z_{s+2}A_j(zaq^4)^{-1} = Z_{s+1}. \tag{47}$$

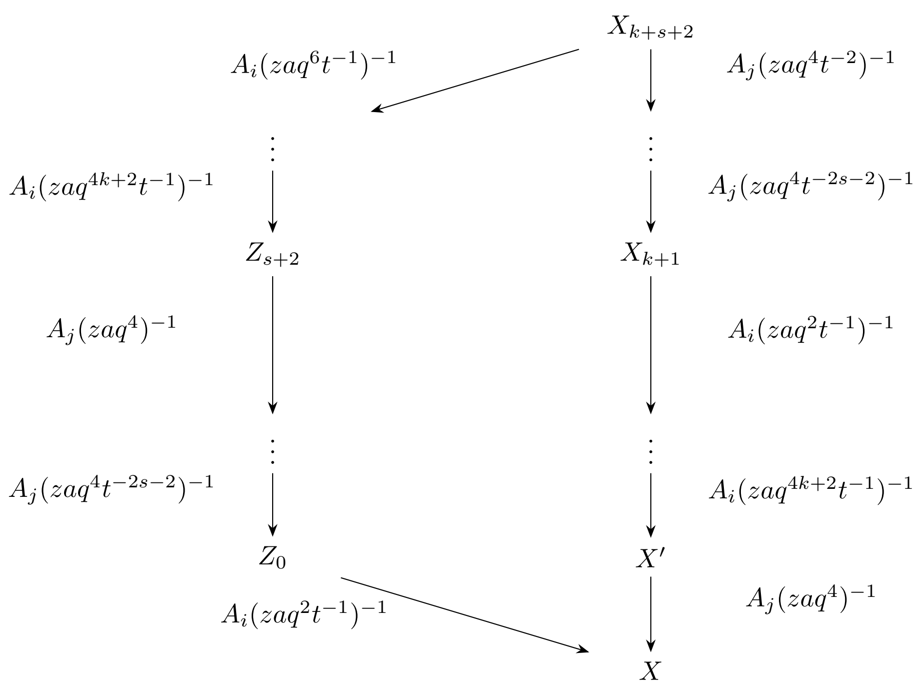
We assume there exists $0 \leq m < s+1$ such that we constructed Z_{u+1} for all $m < u \leq s+1$ such that :

$$\forall m < u \leq s+1, \quad Z_{u+1}A_j(zaq^4t^{-2(s+1-u)})^{-1} = Z_u.$$

We want to prove that $Y_j(zaq^5t^{-2(s-m+1)-1})$ is admissible in Z_{m+1} so that the algorithm gives the right monomial Z_m.
It is clear that Z_{m+1} does not contain $Y_j(zaq^5t^{-2(s-m+1)+1})$ as it has been removed at the previous step.
Moreover, if Z_{m+1} contains $Y_j(zaq^3t^{-2(s-m+1)-1})$ then by construction, X_{k+s+2} also contains $Y_j(zaq^3t^{-2(s-m+1)-1})$. According to the graph, this implies that X_{k+m+2} contains $Y_j(zaq^3t^{-2(s-m+1)-1})$. However, we read in the graph that $Y_j(zaq^5t^{-2(s-m+1)-1})$ is admissible in X_{k+m+2}. It is absurd. Hence, $Y_j(zaq^5t^{-2(s-m+1)-1})$ is admissible in Z_{m+1} and the algorithm gives a monomial Z_m such that :

$$Z_{m+1}A_j(zaq^4t^{-2(s-m+1)})^{-1} = Z_m.$$

Finally, let us prove that $Y_i(zaq^4t^{-2})$ is admissible in Z_0.
We have the equality $Z_0A_i(zq^2t^{-1})^{-1} = X$. Then, by identification of the degrees the admissibility is clear.
Hence the admissibility, and the following graph :



Now, we assume $A_j(zbq^{r_j}t^{-1})$ simplifies $Y_i(zat^2)$.

We have $d_{i,at^{-2}}(X') = -1$. We know that $d_{i,at^{-2}}(X) = 0$, then it implies

$$d_{i,at^{-2}}(A_j(zbqt^{-1})^{-1}) = 1.$$

Hence, $bqt^{-1} = at^{-2}$ and hence

$$b = aq^{-1}t^{-1}.$$

As in the previous case, we want to apply the induction hypothesis and get the following graph for a k defined above :

$$\begin{array}{cl} X_{k+1} & \\ \downarrow & A_i(zaq^2t^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaq^2t^{-2k-1})^{-1} \\ X' & \\ \downarrow & A_j(zat^{-2})^{-1} \\ X & \end{array}$$

Let $k > 0$ the maximal integer such that $Y_i(zat^{-2s})^{-1}$ appears in X' for all $0 \leq s \leq k$.
By definition, $Y_i(zat^{-2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^4t^{-2k})^{-1}$ appears in X', then $Y_i(zat^{-2(k-1)})^{-1}Y_i(zaq^4t^{-2k})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zat^{-2k})^{-1}Y_i(zaq^4t^{-2(k-1)})^{-1}$ appears in X'.
Thus, X' contains $Y_i(zat^{-2(k-2)})^{-1}Y_i(zaq^4t^{-2(k-1)})^{-1}$. We can iterate this reasoning until we get that $Y_i(zaq^4)^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbqt^{-1})^{-1} = A_j(zat^{-2})^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1 A_i(zaq^2t^{-(2k+1)})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_s A_i(zaq^2t^{-(2k-2s+3)})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zat^{-2u})^{-1}$ appears in X_s for all $0 \leq u \leq k+1-s$ and $Y_i(zaq^4t^{-2(u-1)})^{-1}$ does not appear in X_s nor $Y_i(zat^{-2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1} A_i(zaq^2t^{-1})^{-1} = X_k.$$

Moreover,

$$X_{k+1} A_i(zaq^2t^{-1})^{-1} A_i(zaq^2t^{-3})^{-1} \cdots A_i(zaq^2t^{-(2k+1)})^{-1} = X'. \tag{48}$$

By definition of the algorithm, the arrow $X' \to X$ implies that $Y_j(zaqt^{-3})$ is admissible in X'. In particular, it implies that $d_{j,aqt^{-3}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (48) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aqt^{-3}}(X') = d_{j,aqt^{-3}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aqt^{-3}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$ and does not contain $Y_j(zaqt^{-3})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1} \dots Y_j(zaq^{2s+1}t^{-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^{u+1}t^{-3})^{-1}$, then it contains the monomial $Y_j(zaq^{2(u-1)+1}t^{-1})^{-1}Y_j(zaq^{2u+1}t^{-3})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^{2(u-1)+1}t^{-3})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaqt^{-3})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^{2(s-u)+2}t^{-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{gathered} X_{k+s+2}A_j(zaq^2t^{-2})^{-1} \cdots A_j(zaq^{2s+2}t^{-2})^{-1} \times \\ A_i(zaq^2t^{-1})^{-1} \cdots A_i(zaq^2t^{-2k-1})^{-1} A_j(zat^{-2})^{-1} = X. \end{gathered} \tag{49}$$

We get the following path in the graph representation of the algorithm:

$$\begin{array}{cl} X_{k+s+2} & A_j(zaq^2t^{-2})^{-1} \\ \downarrow & \\ \vdots & A_j(zaq^{2s+2}t^{-2})^{-1} \\ \downarrow & \\ X_{k+1} & A_i(zaq^2t^{-1})^{-1} \\ \downarrow & \\ \vdots & A_i(zaq^2t^{-2k-1})^{-1} \\ \downarrow & \\ X' & A_j(zat^{-2})^{-1} \\ \downarrow & \\ X & \end{array}$$

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^4t^{-4}) \dots Y_i(zaq^4t^{-2k-2}) Y_j(zaq^3t^{-3}) \dots Y_j(zaq^{2s+3}t^{-3}). \tag{50}$$

Let us prove that $s > 0$. By contradiction, if $s = 0$ then $Y_i(zaq^4t^{-2})$ is admissible in X_{k+s+2}. Indeed,

$$d_{i,aq^4t^{-2}}(X_{k+s+2}) = d_{i,aq^4t^{-2}}(X_{k+1}) = 1,$$
$$d_{i,aq^4}(X_{k+s+2}) = d_{i,aq^4}(X_{k+1}) = 0,$$
$$d_{i,at^{-2}}(X_{k+s+2}) = d_{i,at^{-2}}(X_{k+1}) = 0.$$

Hence, the algorithm applies the transformation $A_i(zaq^2t^{-1})^{-1}$ to X_{k+s+2} and we get a monomial X^* such that :

$$X_{k+s+2}A_i(zaq^2t^{-1})^{-1} = X^*.$$

Then in particular,

$$d_{j,aqt^{-1}}(X^*) = d_{j,aqt^{-1}}(X_{k+s+2}) + 1 = d_{j,aqt^{-1}}(X_{k+s+1}) + 1 + 1 = 1.$$

Indeed, by genericity, $d_{j,aqt^{-1}}(X_{k+s+1})$ has to be equal to -1. Moreover, by admissibility of $Y_j(zaq^3t^{-3})$ in X_{k+s+2} and regularity,

$$d_{j,aq^3t^{-3}}(X^*) = d_{j,aq^3t^{-3}}(X_{k+s+2}) = 1,$$

$$d_{j,aqt^{-3}}(X^*) = d_{j,aqt^{-3}}(X_{k+s+2}) = 0,$$

Then X^* contains a term of the form $Y_j(zc)Y_j(zcq^{-2r_j}t^2)$ but does not contain $Y_j(zcq^{-2r_j})$ (with $c = q^3t^{-3}$) and X^* is not regular. It is absurd. Hence, $s > 0$.

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaq^2t^{-1})^{-1}$ in order to conclude.
Firstly,

$$d_{i,aq^4t^{-4}}(X_{k+s+2}) = d_{i,aq^4t^{-4}}(X_k) = 1,$$

because as shown in the graph below, $Y_i(zaq^4t^{-4})$ is admissible in X_k. Moreover, $d_{i,at^{-4}}(X_{k+s+2}) = d_{i,at^{-4}}(X_k) = 0$ for the same argument of admissibility. Additionally, if $d_{i,aq^4t^{-2}}(X_{k+s+2}) = 1$ then by definition of $A_j(zaq^4t^{-2})^{-1}$ it implies $d_{i,aq^4t^{-2}}(X_{k+s}) = 2$ which is impossible by the genericity hypothesis.
Hence, $Y_i(zaq^4t^{-4})$ is admissible in X_{k+s+2}.

We construct

$$Z_{k+s+1} := X_{k+s+2}A_i(zaq^2t^{-3})^{-1}.$$

We assume there exists $1 \leq m \leq k$ such that we constructed Z_{s+u+1}, for $m < u \leq k$ such that

$$Z_{u+s+1} = Z_{u+s+2}A_i(zaq^2t^{-2(k-u)-3})^{-1}.$$

Let us construct Z_{s+m+1}.
Firstly, according to the graph,

$$d_{i,aq^4t^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,aq^4t^{-2(k-m)-4}}(X_m) = 1.$$

Moreover,

$$d_{i,aq^4t^{-2(k-m)-2}}(Z_{s+m+2}) = d_{i,aq^4t^{-2(k-m)-2}}(Z_{s+m+3}) - 1 = d_{i,aq^4t^{-2(k-m)-2}}(X_{m+1}) - 1 = 0,$$

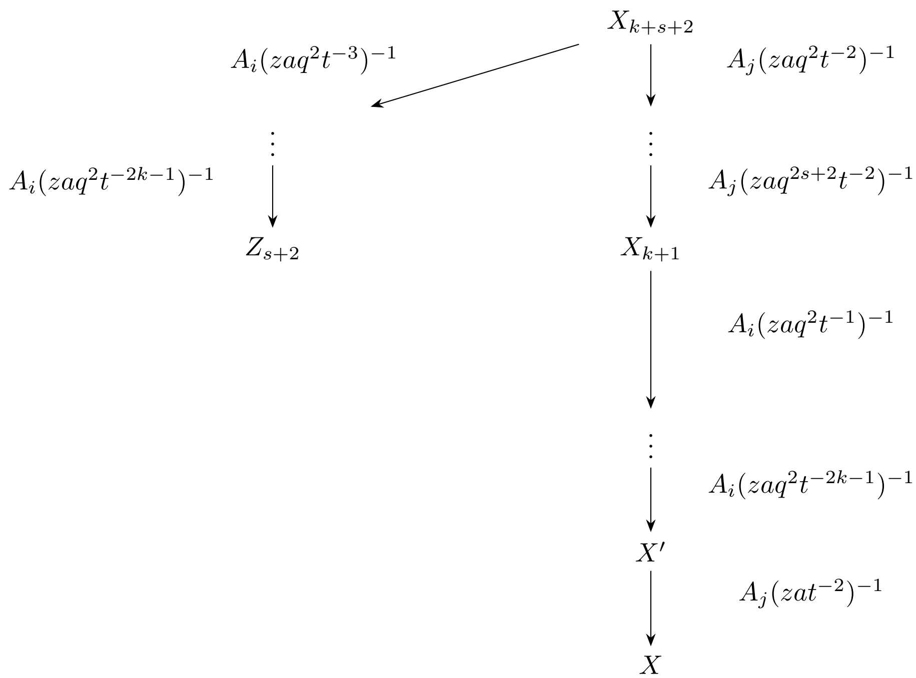
because of the transformation $Z_{s+m+3} \to Z_{s+m+2}$ and because $Y_i(zaq^4t^{-2(k-m)-2})$ is admissible in X_{m+1}. Additionally, $Y_i(zaq^4t^{-2(k-m)-4})$ is admissible in X_m. Thus,

$$d_{i,at^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,at^{-2(k-m)-4}}(X_{k+s+2}) = d_{i,at^{-2(k-m)-4}}(X_m) = 0.$$

Hence the admissibility of $Y_i(zaq^4t^{-2(k-m)-4})$ in Z_{s+m+2} and the existence of the following transformation given by the algorithm :

$$Z_{m+s+1} = Z_{m+s+2}A_i(zaq^2t^{-2(k-m)-3})^{-1}.$$

We obtain the following graph :



Again, we want to construct a path from Z_{s+2} to X in the algorithm graph.
Firstly, by definition of the variables A_i and by admissibility in X',

$$\begin{aligned} d_{j,aqt^{-3}}(Z_{s+2}) &= d_{j,aqt^{-3}}(X_{k+s+2}) + 1, \\ &= d_{j,aqt^{-3}}(X_k) + 1, \\ &= d_{j,aqt^{-3}}(X_{k-1}) - 1 + 1, \\ &= d_{j,aqt^{-3}}(X') = 1. \end{aligned}$$

Moreover,

$$d_{j,aq^{-1}t^{-3}}(Z_{s+2}) = d_{j,aq^{-3}t^{-3}}(X') = 0,$$

as $Y_j(zaqt^{-3})$ is admissible in X'. Additionally, we deduce from the graph the following computation

$$\begin{aligned} d_{j,aqt^{-1}}(Z_{s+2}) &= d_{j,aqt^{-1}}(X_{k+s+2}) \\ &= d_{j,aqt^{-1}}(X_{k+s+1}) - 1 \\ &= d_{j,aqt^{-1}}(X_k) + 1 - 1 \\ &= d_{j,aqt^{-1}}(X') = 0 \end{aligned}$$

Hence, $Y_j(zaqt^{-3})$ is admissible in Z_{s+2}. Thus, the algorithm constructs the following monomial :

$$Z_{s+1} = Z_{s+2}A_j(zat^{-2})^{-1}.$$

Now, we assume there exists $0 \leq m \leq s$ such that we constructed Z_{u+1}, for $m \leq u \leq s$ such that

$$Z_{u+1} = Z_{u+2}A_j(zaq^{2(s-u)}t^{-2})^{-1}.$$

Let us construct Z_m.
Let us prove that $Y_j(zaq^{2(s+1-m)+1}t^{-3})$ is admissible in Z_{m+1}.
Firstly, according to the graph and by admissibility,

$$d_{j,aq^{2(s+1-m)+1}t^{-3}}(Z_{m+1}) = d_{j,aq^{2(s+1-m)}t^{-3}}(X_{k+m+2}) = 1.$$

Moreover,

$$d_{j,aq^{2(s-m)+1}t^{-3}}(Z_{m+1}) = d_{j,aq^{2(s-m)+1}t^{-3}}(Z_{m+2}) - 1 = 0,$$

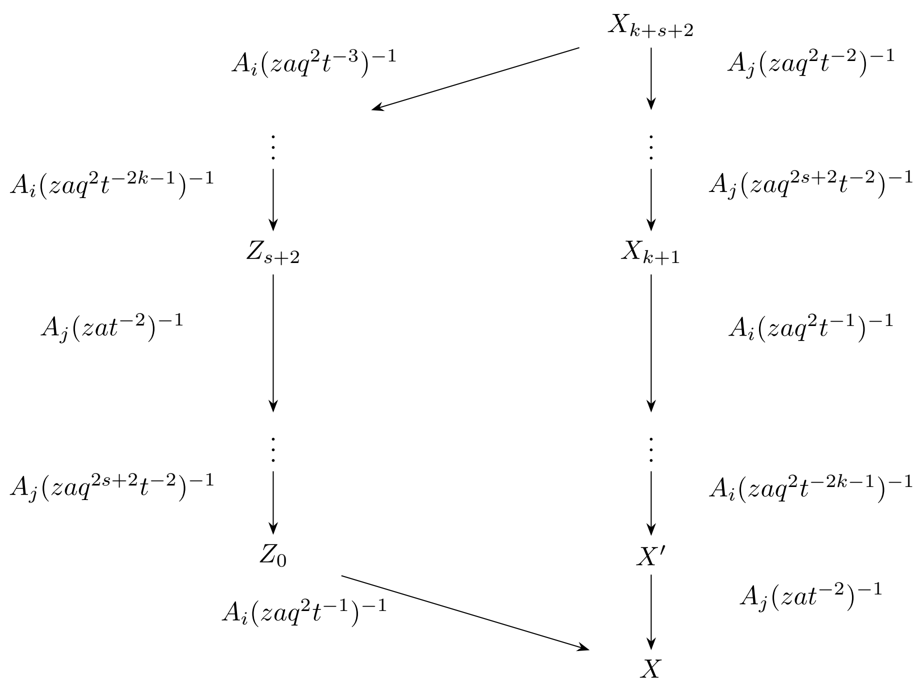
by admissibility and because of the transformation $Z_{m+2} \to Z_{m+1}$.
Additionally, $Y_j(zaq^{2(s+1-m)+1}t^{-3})$ is admissible in X_{k+m+2}. Thus,

$$d_{j,aq^{2(s+1-m)+1}t^{-1}}(Z_{m+1}) = d_{j,aq^{2(s+1-m)+1}t^{-1}}(X_{k+m+2}) = 0.$$

Hence the admissibility of $Y_j(zaq^{2(s+1-m)+1}t^{-3})$ in Z_{m+1} and the existence of the following transformation given by the algorithm :

$$Z_m = Z_{m+1}A_j(zaq^{2(s+1-m)}t^{-2})^{-1}.$$

To conclude, we have to prove that $Y_i(zaq^4t^{-2})$ is admissible in Z_0.
We have the equality $Z_0A_i(zq^2t^{-1})^{-1} = X$. Then, by identification of the degrees the admissibility is clear.
Hence the following paths of transformations in the algorithm :



b) We assume there exists a monomial X'' also appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X',$$

then X'' does contain $Y_i(zaq^{2r_i}t^{-2})$.

Here there are two cases to consider :

- $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_i(zaq^{2r_i}t^{-2})$ comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$.
- X'' contains $Y_j(zaq^{2r_j}t^{-2})$.

Firstly, we assume $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_j(zaq^{2r_j}t^{-2})$ in X' comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$. Hence, $i \neq j$, and we assume the Dynkin subdiagram with nodes $\{i, j\}$ is of type B_2 with $(i, j) = (2, 1)$.
Thus,

$$bq^2t^{-2} = aqt^{-1} \qquad \text{or} \qquad bq^2t^{-2} = aq^3t^{-1}$$

If $bq^2t^{-2} = aqt^{-1}$ then $a = bqt^{-1}$ and $d_{i,a}(A_j(zbqt^{-1})^{-1}) = 1$. However, $d_{i,a}(X) = -1$ and $X'A_j(zbq^{r_j}t^{-1})^{-1} = X$. Thus, $d_{i,a}(X') = -2$ which is absurd because it contradicts the genericity.
Hence, $bq^2t^{-2} = aq^3t^{-1}$. By admissibility, $d_{j,aq^3t^{-1}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$.
But $d_{j,aqt^{-1}}(A_i(zaq^2t^{-1})^{-1}) = 1$. Thus, $d_{j,aqt^{-1}}(X'') = -1$.
Let $s \geq 0$ be the maximal integer such that X'' contains the monomial :

$$Y_j(zaqt^{-1})^{-1} \dots Y_j(zaqt^{-2s-1})^{-1}.$$

If for one $0 \leq u \leq s$, X'' contains $Y_j(zaq^3t^{-2u-1})^{-1}$ by regularity we deduce that X'' contains $Y_j(zaq^3t^{-1})^{-1}$.
But we have :

$$d_{j,aq^3t^{-1}}(X'') = d_{j,aq^3t^{-1}}(X') - d_{j,aq^3t^{-1}}(A_i(zaq^2t^{-1})^{-1}) = 1 - 1 = 0.$$

Hence, it is absurd. Thus, the induction hypoyhesis allow to construct $X_1, \dots, X_{s+1}$ such that we get the following subgraph in the algorithm graph :

$$\begin{array}{cl} X_{s+1} & \\ \downarrow & A_j(zaq^2t^{-2})^{-1} \\ \vdots & \\ \downarrow & \\ X_1 & \\ \downarrow & A_j(zaq^2t^{2s-2})^{-1} \\ X'' & \\ \downarrow & A_i(zaq^2t^{-1})^{-1} \\ X' & \\ \downarrow & A_j(zaq^2)^{-1} \\ X & \end{array}$$

In particular, by admissibility and regularity,

$$d_{j,aq^3t^{-3}}(X_{s+1}) = 1, \qquad d_{j,aqt^{-1}}(X_{s+1}) = 0, \qquad d_{j,aqt^{-3}}(X_{s+1}) = 0.$$

Now, $d_{i,c}(X_{s+1}) = d_{i,c}(X'')$ for $c = aq^4t^{-2}, aq^4, at^{-2}$. Hence, the admissibility of $Y_i(zaq^4t^{-2})$ in X'' implies its admissibility in X_{s+1}, and the creation of

$$Z = X_{s+1}A_i(zaq^2t^{-1})^{-1}$$

in the algorithm. Hence, $d_{j,aqt^{-1}}(Z) = d_{j,aq^3t^{-3}}(Z) = 1$. and $d_{j,aqt^{-3}}(Z) = 0$. This contradicts the regularity of Z.

Hence, we are in the second case and X'' contains $Y_j(zaq^{2r_j}t^{-2})$.
Let us prove that $Y_j(zbq^{2r_j}t^{(}-2))$ is admissible in X''.
If $i = j$ (so that $r_i = r_j$), then by contradiction we assume that $Y_j(zbq^{2r_j}t^{-2})$ is not admissible in X''. We know that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X' so that the presence of $Y_i(zaq^{2r_i}t^{-2})$ in X'' prevents from doing the transformation $A_j(bq^{r_j}t^{-1})^{-1}$. It implies that $a = bq^{-2r_i}$ or $a = bt^2$. This implies X contains $Y_i(zb)^{-1} = Y_i(zaq^{2r_i})^{-1}$ or $Y_i(zat^{-2})^{-1}$. It is absurd by the initial assumption on X. Thus, $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i \neq j$ then it is clear that the admissibility of $Y_j(zbq^{2r_j}t^{(}-2))$ in X' implies its admissibility in X''.
Finally, there exists a monomial

$$X_{new} = X''A_j(zaq^{r_j}t^{-1})^{-1}$$

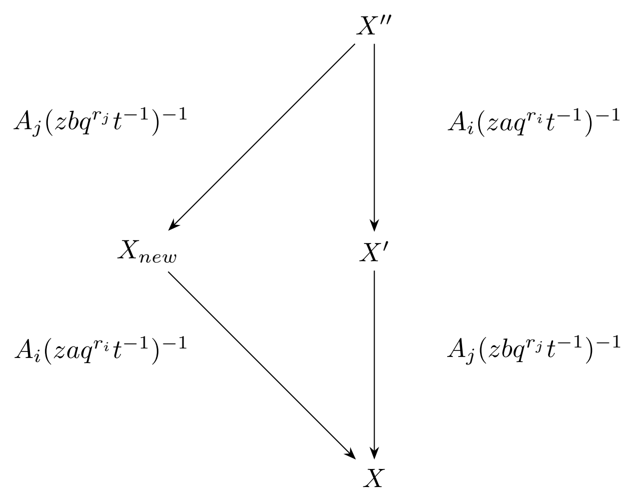
given by the algorithm and X_{new} contains $Y_i(zaq^{2r_i}t^{-2})$. Moreover, if the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$ is not admissible from X_{new} then there exists a factor $Y_i(zat^{-2})$ (resp. $Y_i(zaq^{2r_i})$) blocking this transformation. But this would imply that this factor also appears in X as we have the following equality of monomials :

$$X_{new} = A_i(zaq^{r_i}t^{-1})^{-1}X.$$

We remark that a priori, this equality is not sufficient to get an arrow from X_{new} to X. This equality implies :

$$d_{i,aq^{2r_i}}(X_{new}) = d_{i,aq^{2r_i}}(X),$$

and same for at^{-2}. And hence X does contain $Y_i(zc)$ and $Y_i(zct^2)^{-1}$ with $c = at^{-2}$ (resp. $Y_i(zcq^{-2r_i})^{-1}$ with $c = aq^{2r_i}$) which contradicts the regularity of X. It is impossible. Hence the admissibility of $Y_i(zaq^{2r_i}t^{-2})$ in X_{new}.
Finally we get the following paths :



This concludes the proof in type B_2.

□

B.2. **In type G_2:** We have $r_1 = 3$, $r_2 = 1$,

$$A_1(z) =: Y_1(zq^{-3}t)Y_1(zq^3t^{-1})Y_2(zq^{-2})^{-1}Y_2(z)^{-1}Y_2(zq^2)^{-1} :$$

$$A_2(z) =: Y_2(zq^{-1}t)Y_2(zqt^{-1})Y_1(z)^{-1} :$$

B.2.1. *In the case $(i, j) = (2, 1)$:*

Proof. We prove it by induction on the height of the monomials. We assume the Lie subalgebra generated by the nodes i and j is of type B_2, and $(i, j) = (1, 2)$. *Height 0 :* The only monomial with height 0 is the dominant generic monomial from which we start our algorithm. Hence the property at height 0.

Height $h + 1$: We assume the property is true for all the monomials with heights $h' \leq h \in \mathbb{N}$. Let us prove it for the monomials with height $h + 1$. Let X be a monomial with height $h + 1$. The monomial appears in the algorithm, so it has to come from a monomial X' with height h. Let $A_j(zbq^{r_j}t^{-1})^{-1}$ be the transformation involved. It implies that $d_{j,b}(X) = -1$ and $d_{j,bq^{2r_j}t^{-2}}(X') = 1$. Let $i \in I$, $a \in K$ such that $d_{i,a}(X) = -1$ and $d_{i,aq^{2r_i}}(X), d_{i,at^{-2}}(X) \neq -1$. If $(i, a) = (j, b)$ then we have the result. Else, we have

$$X = X'A_j(zbq^{r_j}t^{-1})^{-1}.$$

Hence,

$$\begin{cases} d_{j,bq^{2r_j}t^{-2}}(X') = 1 \\ d_{j,b}(X) = -1 \\ d_{j,c}(X) = d_{j,c}(X') \text{ for all } c \notin \{b, bq^{2r_j}t^{-2}\} \\ d_{k,c}(X) \geq d_{k,c}(X') \text{ for all } k \neq j,\ c \in K \end{cases}$$

We know that $d_{i,a}(X) = -1$, and $(i, a) \neq (j, b)$. It implies $d_{i,a}(X') \leq -1$. The monomials are all generic, thus

$$d_{i,a}(X') = -1$$

Then by the induction assumption, one of the two following assertions is true :

(a) $d_{i,aq^{2r_i}}(X') = -1$ or $d_{i,at^{-2}}(X') = -1$.
(b) There exists a monomial X'' appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X'.$$

a) Firstly we assume (a) is true and X' does contain $Y_i(zaq^{2r_i})^{-1}$ (i.e $d_{i,aq^{2r_i}}(X') = -1$). Let us prove that this case is absurd, then we will consider the case where X' does contain $Y_i(zat^2)^{-1}$ (i.e $d_{i,at^2}(X') = -1$) By assumption, $d_{i,aq^{2r_i}}(X) = 0$. It implies that the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ simplifies $Y_i(zaq^{2r_i})^{-1}$ and in particular $i \neq j$. Therefore, the expressions depend on the type of the Dynkin diagram generated by the nodes i and j.

Let us do it in type G_2, $(i, j) = (2, 1)$.

$$\begin{aligned} A_j(zbq^{r_j}t^{-1})^{-1} &= A_j(zbq^3t^{-1})^{-1} \\ &= Y_j(zbq^6t^{-2})^{-1}Y_j(zb)^{-1}Y_i(zbqt^{-1})Y_i(zbq^3t^{-1})Y_i(zbq^5t^{-1}) \end{aligned}$$

Then

$$bqt^{-1} = aq^2, \qquad \text{or} \qquad bq^3t^{-1} = aq^2, \qquad \text{or} \qquad bq^5t^{-1} = aq^2.$$

In the two last cases, the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ also simplifies $Y_i(za)^{-1}$. This is impossible. Thus, we are in the first case and as in types A_2, B_2 we get :

$$b = aqt.$$

Let $k > 0$ the maximal integer such that $Y_i(zaq^{2s})^{-1}$ appears in X' for all $0 \leq s \leq k$. We want to prove that the graph representation of the algorithm contains the following subgraph :

$$\begin{array}{cc}
X_{k+1} & \\
\downarrow & A_i(zaqt^{-1})^{-1} \\
\vdots & \\
\downarrow & A_i(zaq^{2k+1}t^{-1})^{-1} \\
X' & \\
\downarrow & A_j(zaq^4)^{-1} \\
X &
\end{array}$$

By definition, $Y_i(zaq^{2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^{2k}t^{-2})^{-1}$ appears in X', then $Y_i(zaq^{2(k-1)})^{-1}Y_i(zaq^{2k}t^{-2})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zaq^{2k})^{-1}Y_i(zaq^{2(k-1)}t^{-2})^{-1}$ appears in X'. Thus, X' contains $Y_i(zaq^{2(k-2)})^{-1}Y_i(zaq^{2(k-1)}t^{-2})^{-1}$. We can iterate this reasoning until we get that $Y_i(zat^{-2})^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^{r_j}t^{-1})^{-1} = A_j(zaq^4)^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1 A_i(zaq^{2k+1}t^{-1})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_s A_i(zaq^{2k-2s+3}t^{-1})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zaq^{2u})^{-1}$ appears in X_s for all $0 \leq u \leq k+1-s$ and $Y_i(zaq^{2(u-1)}t^{-2})^{-1}$ does not appear in X_s nor $Y_i(zaq^{2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1} A_i(zaqt^{-1})^{-1} = X'.$$

But $d_{j,aqt^{-1}}(A_i(zaqt^{-1})^{-1}) = 1$. Thus,

$$d_{j,aqt^{-1}}(X_{k+1}) = d_{j,aqt^{-1}}(X_k) - 1 = d_{j,aqt^{-1}}(X') - 1 < 0$$

because by regularity,

$$d_{j,aqt^{-1}}(X') \in \{0, 1\},$$

and if $d_{j,aqt^{-1}}(X') = 1$ then the transformation $A_j(zbq^3t^{-1})^{-1} = A_j(zaq^4)^{-1}$ would not have been admissible from X'. Thus $d_{j,aqt^{-1}}(X') = 0$ and X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$. Moreover,

$$X_{k+1} A_i(zaqt^{-1})^{-1} A_i(zaq^3t^{-1})^{-1} \cdots A_i(zaq^{2k+1}t^{-1})^{-1} = X'. \tag{51}$$

We recall that $X'A_j(zaq^4)^{-1} = X$. By definition of the algorithm, it implies $Y_j(zaq^7t^{-1})$ is admissible in X'. In particular, it implies that $d_{j,aq^7t^{-1}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (51) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aq^7t^{-1}}(X') \leq d_{j,aq^7t^{-1}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aq^7t^{-1}}(X_{k+1}) \geq 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$ and does not contain $Y_j(zaq^7t^{-1})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1} \dots Y_j(zaqt^{-2s-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^7t^{-2u-1})^{-1}$, then it contains the monomial $Y_j(zaqt^{-2u+1})^{-1}Y_j(zaq^7t^{-2u-1})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^7t^{-2u+1})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaq^7t^{-1})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^4t^{-2(s-u)-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{aligned} &X_{k+s+2}A_j(zaq^4t^{-2})^{-1} \cdots A_j(zaq^4t^{-2s-2})^{-1} \times \\ &A_i(zaqt^{-1})^{-1} \cdots A_i(zaq^{2k+1}t^{-1})^{-1} A_j(zaq^4)^{-1} = X. \end{aligned} \tag{52}$$

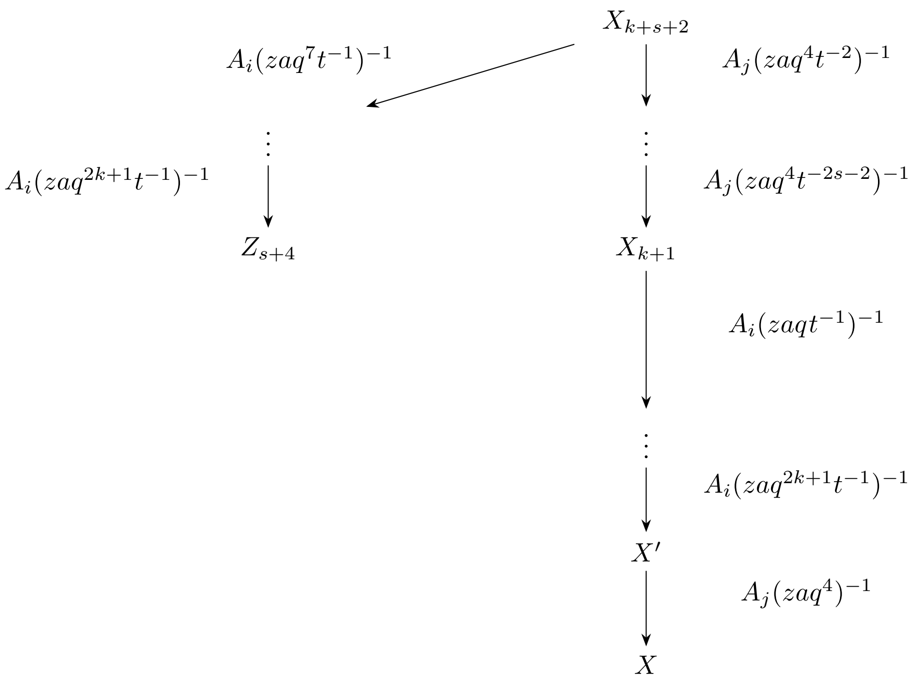
We get the following path in the graph representation of the algorithm:

$$\begin{array}{cl} X_{k+s+2} & \\ \downarrow & A_j(zaq^4t^{-2})^{-1} \\ \vdots & \\ \downarrow & A_j(zaq^4t^{-2s-2})^{-1} \\ X_{k+1} & \\ \downarrow & A_i(zaqt^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaq^{2k+1}t^{-1})^{-1} \\ X' & \\ \downarrow & A_j(zaq^4)^{-1} \\ X & \end{array}$$

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^8t^{-2}) \dots Y_i(zaq^{2k+2}t^{-2}) Y_j(zaq^7t^{-3}) \dots Y_j(zaq^7t^{-2s-3}). \tag{53}$$

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaqt^{-1})^{-1}$ in order to conclude.
We want to obtain the following subgraph :



Firstly, let us prove that $k \geq 3$. The transformation $X' \to X$ gives that $Y_j(zaq^7t^{-1})$ is admissible in X'. Thus, $d_{j,aq^7t^{-1}}(X') = 1$. By contradiction, if $k < 3$ then from the graph below we deduce that

$$d_{j,aq^7t^{-1}}(X') = d_{j,aq^7t^{-1}}(X_{k+s+2}) = 1.$$

Hence, $Y_j(zaq^7t^{-3})$ is not admissible in X_{k+s+2}, which is absurd. Hence, $k \geq 3$.

To apply the transformation $A_i(zaq^7t^{-1})^{-1}$, we need to check that $Y_i(zaq^8t^{-2})$ is admissible in X_{k+s+2}. We know that X_{k+s+2} does not contain $Y_i(zaq^6t^{-2})$ as it would imply that X_{k+s+1} contains $Y_i(zaq^6t^{-2})^2$ which is absurd as all monomials are generic by assumption. Moreover, if X_{k+s+2} contains $Y_i(zaq^8)$ then the equation (52) implies :

$$d_{i,aq^8}(X_{k+s+2}) = d_{i,aq^8}(X_{k-2}) = 1.$$

But the graph gives that $Y_i(zaq^8t^{-2})$ is admissible in X_{k-2}. It is absurd. Thus, $Y_i(zaq^8t^{-2})$ is admissible in X_{k+s+2} and the algorithm apply the transformation $A_i(zaq^7t^{-1})^{-1}$ to X_{k+s+2}, giving a monomial Z_{k+s+1} such that :

$$X_{k+s+2}A_i(zaq^7t^{-1})^{-1} = Z_{k+s+1}. \tag{54}$$

We assume there exists $3 \leq m \leq k-1$ such that the algorithm gives Z_{u+s+1} for all $m < u \leq k$ such that :

$$\forall m < u \leq k, \quad Z_{u+s+2}A_i(zaq^{2(k-u+3)+1}t^{-1})^{-1} = Z_{u+s+1},$$

setting $Z_{k+s+2} := X_{k+s+2}$. We want to prove that the algorithm gives Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{2(k-m+3)+1}t^{-1})^{-1} = Z_{m+s+1}.$$

We know that

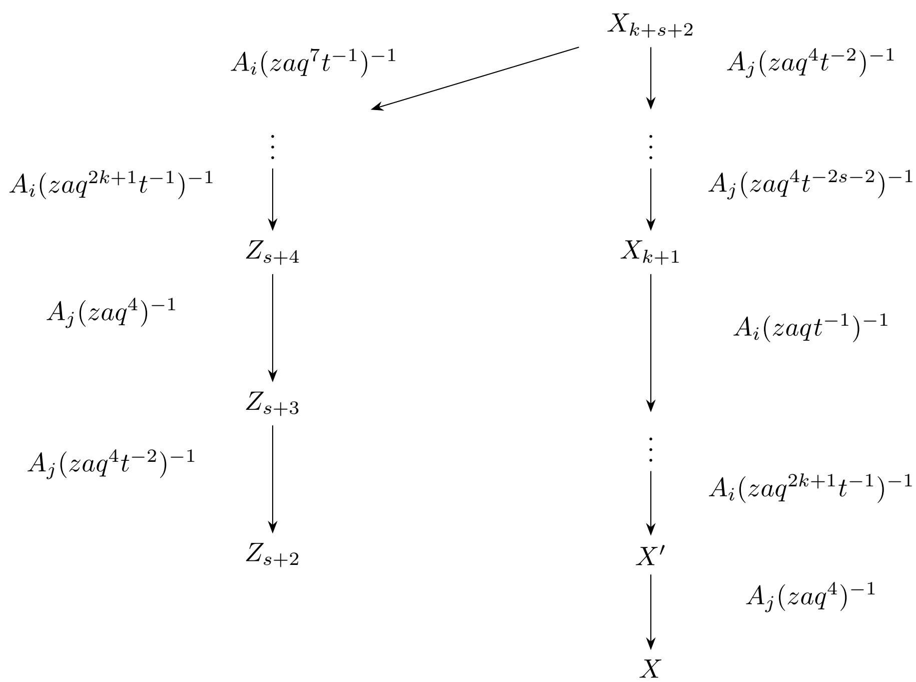
$$Z_{m+s+2} = X_{k+s+2} A_i(zaq^7 t^{-1})^{-1} \cdots A_i(zaq^{2(k-m+2)+1} t^{-1})^{-1}. \tag{55}$$

Then, Z_{m+s+2} contains $Y_i(zaq^{2(k-m+4)}t^{-2}) \ldots Y_i(zaq^{2k+2}t^{-2})$.
Let us check that $Y_i(zaq^{2(k-m+4)}t^{-2})$ is admissible in Z_{m+s+2}.
It is clear that Z_{m+s+2} does not contain $Y_i(zaq^{2(k-m+3)}t^{-2})$ as it has been removed at the previous step. Moreover, if Z_{m+s+2} contains $Y_i(zaq^{2(k-m+4)})$. then by (55), X_{k+s+2} contains $Y_i(zaq^{2(k-m+4)})$. Thus, according to the graph, X_{m-2} contains $Y_i(zaq^{2(k-m+4)})$. However, we read in the same figure that $Y_i(zaq^{2(k-m+3)}t^{-2})$ is admissible in X_{m-2}. It is absurd. Hence, $Y_i(zaq^{2(k-m+4)}t^{-2})$ is admissible in Z_{m+s+2} and the algorithm gives a monomial Z_{m+s+1} such that :

$$Z_{m+s+2} A_i(zaq^{2(k-m+3)+1} t^{-1})^{-1} = Z_{m+s+1}.$$

By induction, we get monomials $Z_{s+4}, \ldots, Z_{k+s+1}$ verifiying the equation (55).

Now we want to complete our graph to the following :



Furthermore, by (54), we get :

$$d_{j,aq^7t^{-1}}(Z_{s+4}) = d_{j,aq^7t^{-1}}(Z_{k+s+1}) = d_{j,aq^7t^{-1}}(X_{k+s+2}) + 1.$$

But we know by admissibility of $Y_j(zaq^7t^{-3})$ and regularity of X_{k+s+2} that :

$$d_{j,aq^7t^{-1}}(X_{k+s+2}) = 0.$$

Hence, $d_{j,aq^7t^{-1}}(Z_{s+4}) = 1$.

Thus, the monomial Z_{s+4} contains $Y_j(zaq^7t^{-1})\dots Y_j(zaq^7t^{-2s-3})$. We want to check that $Y_j(zaq^7t^{-1})$ is admissible in Z_{s+4}.
If Z_{s+4} contains $Y_j(zaqt^{-1})$ then it contains $Y_j(zaq^7t^{-3})Y_j(zaqt^{-1})$ (this is of the form $Y_j(zc)Y_j(zcq^{-2r_j}t^2)$). By regularity, it also contains $Y_j(zaqt^{-3})$. By (55), X_{k+s+2} contains $Y_j(zaqt^{-3})$, and $Y_j(zaq^7t^{-3})$ is not admissible in X_{k+s+2}. It is absurd.
If Z_{s+4} contains $Y_j(zaq^7t)$ then by (55), X_{k+s+2} contains $Y_j(zaq^7t)$. According to the graph, this implies that X' contains $Y_j(zaq^7t)$. But $Y_j(zaq^7t^{-1})$ is admissible in X'. It is absurd.
Hence, $Y_j(zaq^7t^{-1})$ is admissible in Z_{s+4} and the algorithm gives a monomial Z_{s+3} such that :

$$Z_{s+4}A_j(zaq^4)^{-1} = Z_{s+3}. \tag{56}$$

Now we want to construct Z_{s+2} such that

$$Z_{s+3}A_j(zaq^4t^{-2})^{-1} = Z_{s+2}.$$

Let us prove that $Y_j(zaq^7t^{-3})$ is admissible in Z_{s+3}.
By admissibility,

$$d_{j,aq^7t^{-3}}(Z_{s+3}) = d_{j,aq^7t^{-3}}(X_{k+s+2}) = 1.$$

Moreover,

$$d_{j,aqt^{-3}}(Z_{s+3}) = d_{j,aqt^{-3}}(X_{k+s+2}) = 0,$$

and

$$d_{j,aq^7t^{-1}}(Z_{s+3}) = d_{j,aq^7t^{-1}}(Z_{s+4}) - 1 = 1 - 1 = 0.$$

Hence the admissibility and the monomial Z_{s+2} appears in the algorithm.

Finally, let us prove that Z_{s+2} is not regular.
By regularity and because $Y_i(zaq^4t^{-2})$ is admissible in X_k, we have $d_{i,aq^4}(X_k) = 0$. Thus, the graph gives

$$\begin{aligned} d_{i,aq^4}(Z_{s+2}) &= d_{i,aq^4}(Z_{s+4}) + 1, \\ &= d_{i,aq^4}(X_{k+s+2}) + 1, \\ &= d_{i,aq^4}(X_k) + 1, \\ d_{i,aq^4}(Z_{s+2}) &= 1. \end{aligned}$$

Moreover, by admissibility, $d_{i,aq^6t^{-2}}(X_{k-1}) = 1$. Thus,

$$\begin{aligned} d_{i,aq^6t^{-2}}(Z_{s+2}) &= d_{i,aq^6t^{-2}}(Z_{s+3}) + 1, \\ &= d_{i,aq^6t^{-2}}(X_{k+s+2}) + 1, \\ &= d_{i,aq^6t^{-2}}(X_{k+s+1}) - 1 + 1, \\ &= d_{i,aq^6t^{-2}}(X_{k-1}), \\ d_{i,aq^6t^{-2}}(Z_{s+2}) &= 1. \end{aligned}$$

However, by construction of the algorithm, the transformation $X_{k+s+2} \to Z_{k+s+1}$ gives that $d_{i,aq^6}(Z_{k+s+1}) = -1$. Thus the graph implies

$$\begin{aligned} d_{i,aq^6}(Z_{s+2}) &= d_{i,aq^6}(Z_{s+4}) + 1, \\ &= d_{i,aq^6}(Z_{k+s+1}) + 1, \\ d_{i,aq^6}(Z_{s+2}) &= 0. \end{aligned}$$

Hence setting $c = q^6t^{-2}$, Z_1 contains $Y_i(zc)Y_i(zcq^{-2r_i}t^2)$ but does not contain $Y_i(zct^2)$. It is absurd as assumed that the algorithm does not fail.

Now, we assume $A_j(zbq^{r_j}t^{-1})$ simplifies $Y_i(zat^2)$.

Now, we assume $d_{i,at^{-2}}(X') = -1$. We know that $d_{i,at^{-2}}(X) = 0$, then it implies

$$d_{i,at^{-2}}(A_j(zbq^3t^{-1})^{-1}) = 1.$$

Hence,

$$bqt^{-1} = at^{-2} \qquad \text{or} \qquad bq^3t^{-1} = at^{-2} \qquad \text{or} \qquad bq^5t^{-1} = at^{-2}.$$

We assume $bqt^{-1} = at^{-2}$ or $bq^3t^{-1} = at^{-2}$. Firstly, it is clear that $d_{i,a}(X') = -1$ and $d_{i,aq^2}(X') = 0$. Then by regularity, $d_{i,aq^2t^{-2}}(X') \neq -1$. Moreover,

$$d_{i,aq^2t^{-2}}(A_j(zbq^3t^{-1})^{-1}) = 1.$$

Then, $d_{i,aq^2t^{-2}}(X') = d_{i,aq^2t^{-2}}(X) - 1$. Thus, $d_{i,aq^2t^{-2}}(X) = 1$. Hence, X contains $Y_i(zaq^2t^{-2})Y_i(za)^{-1}$. Let us prove that $Y_i(zaq^2t^{-2})$ is admissible in X. If $d_{i,aq^2}(X) = 1$ (resp. $d_{i,at^{-2}}(X) = 1$) then X is not regular as it contains a term of the form $Y_i(zc)Y_i(zcq^{-2r_i})^{-1}$ (resp. $Y_i(zc)Y_i(zct^2)^{-1}$). Hence the admissibility of $Y_i(zaq^2t^{-2})$in X. So, the algorithm implies a transformation to obtain the monomial

$$X^* = XA_izaqt^{-1},$$

and $d_{i,a}(X^*) = -2$ which contradicts the genericity.

Hence, $b = aq^{-5}t^{-1}$. As in the previous case, we want to apply the induction hypothesis and get the following graph for a k defined above :

$$\begin{array}{cl} X_{k+1} & \\ \downarrow & A_i(zaqt^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaqt^{-2k-1})^{-1} \\ X' & \\ \downarrow & A_j(zaq^{-2}t^{-2})^{-1} \\ X & \end{array}$$

Let $k > 0$ the maximal integer such that $Y_i(zat^{-2s})^{-1}$ appears in X' for all $0 \leq s \leq k$.
By definition, $Y_i(zat^{-2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^2t^{-2k})^{-1}$ appears in X', then $Y_i(zat^{-2(k-1)})^{-1}Y_i(zaq^2t^{-2k})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zat^{-2k})^{-1}Y_i(zaq^2t^{-2(k-1)})^{-1}$ appears in X'.
Thus, X' contains $Y_i(zat^{-2(k-2)})^{-1}Y_i(zaq^2t^{-2(k-1)})^{-1}$. We can iterate this reasoning until we get that

$Y_i(zaq^2)^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^3t^{-1})^{-1} = A_j(zaq^{-2}t^{-2})^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1 A_i(zaqt^{-(2k+1)})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_s A_i(zaqt^{-(2k-2s+3)})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zat^{-2u})^{-1}$ appears in X_s for all $0 \leq u \leq k-s$ and $Y_i(zaq^2t^{-2(u-1)})^{-1}$ does not appear in X_s nor $Y_i(zat^{-2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1}A_i(zaqt^{-1})^{-1} = X_k.$$

Thus,

$$X_{k+1}A_i(zaqt^{-1})^{-1}A_i(zaqt^{-3})^{-1}\cdots A_i(zaqt^{-(2k+1)})^{-1} = X'. \tag{57}$$

By definition of the algorithm, the arrow $X' \to X$ implies that $Y_j(zaqt^{-3})$ is admissible in X'. In particular, it implies that $d_{j,aqt^{-3}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (57) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aqt^{-3}}(X') = d_{j,aqt^{-3}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aqt^{-3}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$ and does not contain $Y_j(zaqt^{-3})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1}\ldots Y_j(zaq^{6s+1}t^{-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^{6u+1}t^{-3})^{-1}$, then it contains the monomial $Y_j(zaq^{6(u-1)+1}t^{-1})^{-1}Y_j(zaq^{6u+1}t^{-3})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^{6(u-1)+1}t^{-3})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaqt^{-3})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^{6(s-u)+4}t^{-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{aligned} &X_{k+s+2}A_j(zaq^4t^{-2})^{-1}\cdots A_j(zaq^{6s+4}t^{-2})^{-1}\times \\ &\quad A_i(zaqt^{-1})^{-1}\cdots A_i(zaqt^{-2k-1})^{-1}A_j(zaq^{-2}t^{-2})^{-1} = X. \end{aligned} \tag{58}$$

We get the following path in the graph representation of the algorithm:

$$\begin{array}{cc}
X_{k+s+2} & \\
\downarrow & A_j(zaq^4t^{-2})^{-1} \\
\vdots & \\
\downarrow & A_j(zaq^{6s+4}t^{-2})^{-1} \\
X_{k+1} & \\
\downarrow & A_i(zaqt^{-1})^{-1} \\
\vdots & \\
\downarrow & A_i(zaqt^{-2k-1})^{-1} \\
X' & \\
\downarrow & A_j(zaq^{-2}t^{-2})^{-1} \\
X &
\end{array}$$

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^2t^{-4})\dots Y_i(zaq^2t^{-2k-2})Y_j(zaq^7t^{-3})\dots Y_j(zaq^{6s+7}t^{-3}). \tag{59}$$

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaqt^{-1})^{-1}$ in order to conclude.
Firstly,

$$d_{i,aq^2t^{-4}}(X_{k+s+2}) = d_{i,aq^2t^{-4}}(X_k) = 1,$$

because as shown in the graph below, $Y_i(zaq^2t^{-4})$ is admissible in X_k. Moreover, $d_{i,at^{-4}}(X_{k+s+2}) = d_{i,at^{-4}}(X_k) = 0$ for the same argument of admissibility. Additionally, if $d_{i,aq^2t^{-2}}(X_{k+s+2}) = 1$ then by definition of $A_j(zaq^3t^{-2})^{-1}$ it implies $d_{i,aq^2t^{-2}}(X_{k+s+1}) = 2$ which is impossible by the genericity hypothesis.
Hence, $Y_i(zaq^2t^{-4})$ is admissible in X_{k+s+2}.
We construct

$$Z_{k+s+1} := X_{k+s+2}A_i(zaqt^{-3})^{-1}.$$

We assume there exists $1 \leq m \leq k$ such that we constructed Z_{s+u+1}, for $m < u \leq k$ such that

$$Z_{u+s+1} = Z_{u+s+2}A_i(zaqt^{-2(k-u)-3})^{-1}.$$

Let us construct Z_{s+m+1}.
Firstly, according to the graph,

$$d_{i,aq^2t^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,aq^2t^{-2(k-m)-4}}(X_m) = 1.$$

Moreover,

$$d_{i,aq^2t^{-2(k-m)-2}}(Z_{s+m+2}) = d_{i,aq^2t^{-2(k-m)-2}}(Z_{s+m+3}) - 1 = d_{i,aq^2t^{-2(k-m)-2}}(X_{k+s+2}) - 1 = 0,$$

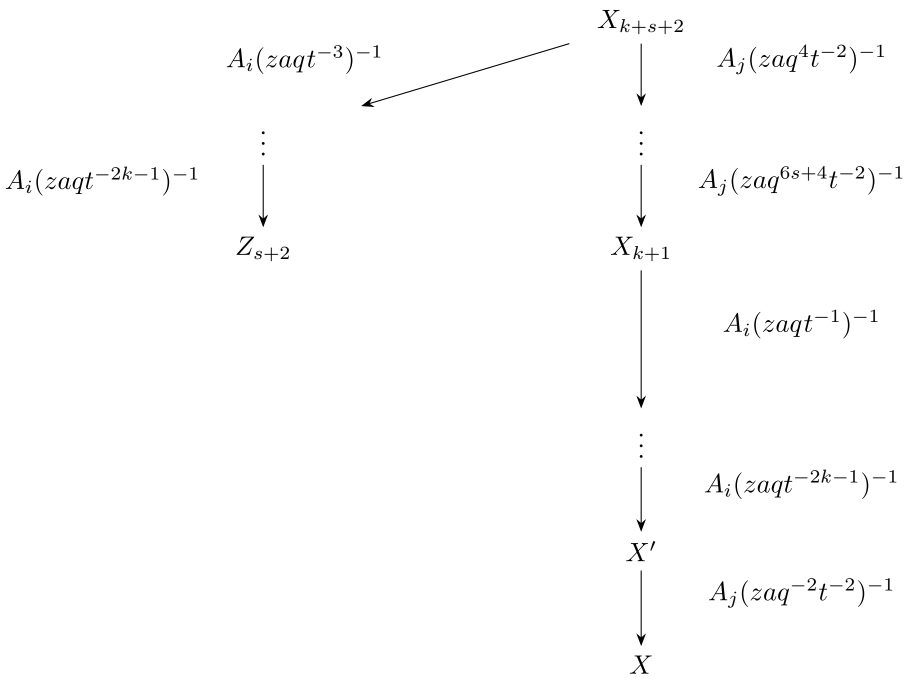
because of the transformation $Z_{s+m+3} \to Z_{s+m+2}$.
Additionally, $Y_i(zaq^2t^{-2(k-m)-4})$ is admissible in X_m. Thus,

$$d_{i,at^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,at^{-2(k-m)-4}}(X_{k+s+2}) = d_{i,at^{-2(k-m)-4}}(X_m) = 0.$$

Hence the admissibility of $Y_i(zaq^2t^{-2(k-m)-4})$ in Z_{s+m+2} and the existence of the following transformation given by the algorithm :

$$Z_{m+s+1} = Z_{m+s+2}A_i(zaqt^{-2(k-u)-3})^{-1}.$$

We obtain the following graph :



Again, we want to construct a path from Z_{s+2} to X in the algorithm graph.
Firstly,

$$\begin{aligned} d_{j,aqt^{-3}}(Z_{s+2}) &= d_{j,aqt^{-3}}(X_{k+s+2}) + 1, \\ &= d_{j,aqt^{-3}}(X_k) + 1, \\ &= d_{j,aqt^{-3}}(X_{k-1}) - 1 + 1, \\ &= d_{j,aqt^{-3}}(X') = 1. \end{aligned}$$

Because $Y_j(zaqt^{-3})$ is admissible in X'.
Moreover,

$$d_{j,aq^{-5}t^{-3}}(Z_{s+2}) = d_{j,aq^{-5}t^{-3}}(X') = 0,$$

as $Y_j(zaqt^{-3})$ is admissible in X'. Additionally, we deduce from the graph the following computation

$$\begin{aligned} d_{j,aqt^{-1}}(Z_{s+2}) &= d_{j,aqt^{-1}}(X_{k+s+2}), \\ &= d_{j,aqt^{-1}}(X_{k+s+1}) + 1, \\ &= d_{j,aqt^{-1}}(X_k) - 1 + 1, \\ &= d_{j,aqt^{-1}}(X') = 0. \end{aligned}$$

Hence, $Y_j(zaqt^{-3})$ is admissible in Z_{s+2}. Thus, the algorithm constructs the following monomial :

$$Z_{s+1} = Z_{s+2}A_j(zaq^{-2}t^{-2})^{-1}.$$

Now, we assume there exists $0 \leq m \leq s$ such that we constructed Z_{u+1}, for $m \leq u \leq s$ such that

$$Z_{u+1} = Z_{u+2} A_j(zaq^{6(s-u)-2}t^{-2})^{-1}.$$

Let us construct Z_m.
Let us prove that $Y_j(zaq^{6(s-m)+7}t^{-3})$ is admissible in Z_{m+1}.
Firstly, according to the graph and by admissibility,

$$d_{j,aq^{6(s-m)+7}t^{-3}}(Z_{m+1}) = d_{j,aq^{6(s-m)+7}t^{-3}}(X_{k+m+2}) = 1.$$

Moreover,

$$d_{j,aq^{6(s-m)+1}t^{-3}}(Z_{m+1}) = d_{j,aq^{6(s-m)+1}t^{-3}}(Z_{m+2}) - 1 = 0,$$

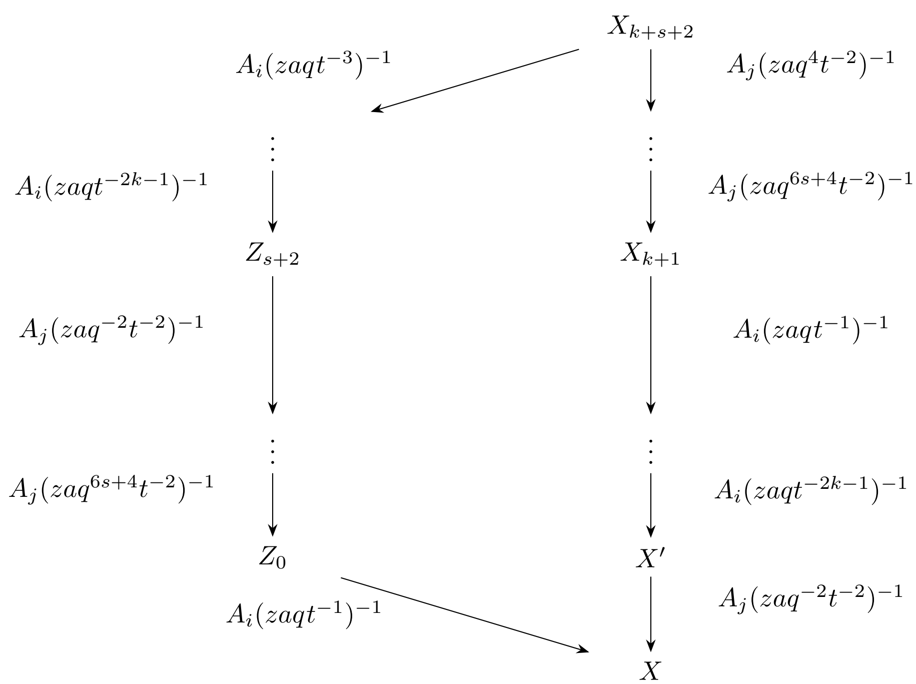
by admissibility and because of the transformation $Z_{m+2} \to Z_{m+1}$.
Additionally, $Y_j(zaq^{6(s-m)+7}t^{-3})$ is admissible in X_{k+m+2}. Thus,

$$d_{j,aq^{6(s-m)+7}t^{-1}}(Z_{m+1}) = d_{j,aq^{6(s-m)+7}t^{-1}}(X_{k+m+2}) = 0.$$

Hence the admissibility of $Y_j(zaq^{6(s-m)+7}t^{-3})$ in Z_{m+1} and the existence of the following transformation given by the algorithm :

$$Z_m = Z_{m+1} A_j(zaq^{6(s-m)+4}t^{-2})^{-1}.$$

To conclude, we have to prove that $Y_i(zaq^2t^{-2})$ is admissible in Z_0.
We have the equality $Z_0 A_i(zqt^{-1})^{-1} = X$. Then, by identification of the respective degrees it is clear.
Hence the following paths of transformations in the algorithm :



b) We assume there exists a monomial X'' also appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X',$$

then X'' does contain $Y_i(zaq^{2r_i}t^{-2})$.

Here there are two cases to consider :

- $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_j(zaq^{2r_j}t^{-2})$ in X' comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$.
- X'' contains $Y_j(zaq^{2r_j}t^{-2})$.

Firstly, we assume $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_i(zaq^{2r_i}t^{-2})$ comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$. Hence, $i \neq j$, and we assume the Dynkin subdiagram with nodes $\{i, j\}$ is of type G_2 with $(i, j) = (2, 1)$. We have $(r_1, r_2) = (1, 3)$. Thus, $bq^6t^{-2} = aqt^{-1}$, and $bq^5t^{-1} = a$. Hence,

$$d_{i,a}(A_j(zbq^3t^{-1})^{-1}) = 1.$$

However, $d_{i,a}(X) = -1$ and $X'A_j(zaq^3t^{-1})^{-1} = X$. Thus, $d_{i,a}(X') = -2$ which contradicts the genericity. It is absurd.

Hence, we are in the second case and X'' contains $Y_j(zaq^{2r_j}t^{-2})$.
Let us prove that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i = j$ (so that $r_i = r_j$), then by contradiction we assume that $Y_j(zbq^{2r_j}t^{-2})$ is not admissible in X''. We know that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X' so that the presence of $Y_i(zaq^{2r_i}t^{-2})$ in X'' prevents from doing the transformation $A_j(bq^{r_j}t^{-1})^{-1}$. It implies that $a = bq^{-2r_i}$ or $a = bt^2$. This implies X contains $Y_i(zb)^{-1} = Y_i(zaq^{2r_i})^{-1}$ or $Y_i(zat^{-2})^{-1}$. It is absurd by the initial assumption on X. Thus, $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i \neq j$ then it is clear that the admissibility of $Y_j(zbq^{2r_j}t^(-2))$ in X' implies its admissibility in X''.
Finally, there exists a monomial

$$X_{new} = X''A_j(zaq^{r_j}t^{-1})^{-1}$$

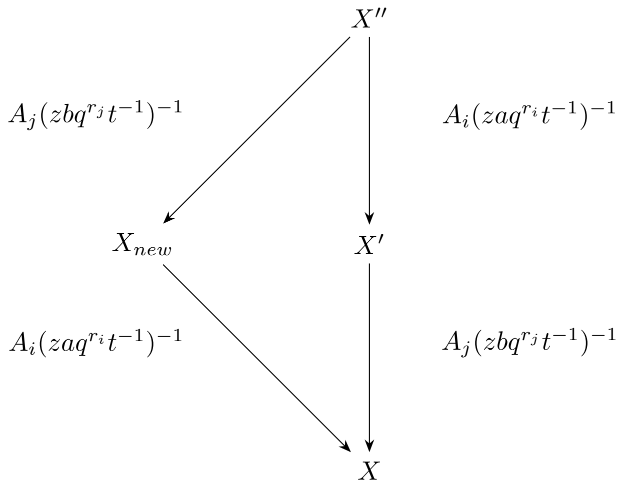
given by the algorithm and X_{new} contains $Y_i(zaq^{2r_i}t^{-2})$. Moreover, if the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$ is not admissible from X_{new} then there exists a factor $Y_i(zat^{-2})$ (resp. $Y_i(zaq^{2r_i})$) blocking this transformation. But this would imply that this factor also appears in X as we have the following equality of monomials :

$$X_{new} = A_i(zaq^{r_i}t^{-1})^{-1}X.$$

We remark that a priori, this equality is not sufficient to get an arrow from X_{new} to X. This equality implies :

$$d_{i,aq^{2r_i}}(X_{new}) = d_{i,aq^{2r_i}}(X),$$

and same for at^{-2}. And hence X does contain $Y_i(zc)$ and $Y_i(zct^2)^{-1}$ with $c = at^{-2}$ (resp. $Y_i(zcq^{-2r_i})^{-1}$ with $c = aq^{2r_i}$) which contradicts the regularity of X. It is impossible. Hence the admissibility of $Y_i(zaq^{2r_i}t^{-2})$ in X_{new}.
Finally we get the following paths :



and hence the result. This concludes the proof in type G_2 for $(i,j) = (2,1)$. ☐

B.2.2. *In the case* $(i,j) = (1,2)$ *:*

Proof. We prove it by induction on the height of the monomials. We assume the Lie subalgebra generated by the nodes i and j is of type B_2, and $(i,j) = (2,1)$. *Height* 0 *:* The only monomial with height 0 is the dominant generic monomial from which we start our algorithm. Hence the property at height 0.

Height $h+1$ *:* We assume the property is true for all the monomials with heights $h' \leq h \in \mathbb{N}$. Let us prove it for the monomials with height $h+1$. Let X be a monomial with height $h+1$. The monomial appears in the algorithm, so it has to come from a monomial X' with height h. Let $A_j(zbq^{r_j}t^{-1})^{-1}$ be the transformation involved. It implies that $d_{j,b}(X) = -1$ and $d_{j,bq^{2r_j}t^{-2}}(X') = 1$. Let $i \in I$, $a \in K$ such that $d_{i,a}(X) = -1$ and $d_{i,aq^{2r_i}}(X), d_{i,at^{-2}}(X) \neq -1$. If $(i,a) = (j,b)$ then we have the result. Else, we have

$$X = X' A_j(zbq^{r_j}t^{-1})^{-1}.$$

Hence,

$$\begin{cases} d_{j,bq^{2r_j}t^{-2}}(X') = 1 \\ d_{j,b}(X) = -1 \\ d_{j,c}(X) = d_{j,c}(X') \text{ for all } c \notin \{b, bq^{2r_j}t^{-2}\} \\ d_{k,c}(X) \geq d_{k,c}(X') \text{ for all } k \neq j,\ c \in K \end{cases}$$

We know that $d_{i,a}(X) = -1$, and $(i,a) \neq (j,b)$. It implies $d_{i,a}(X') \leq -1$. The monomials are all generic, thus

$$d_{i,a}(X') = -1$$

Then by the induction assumption, one of the two following assertions is true :

(a) $d_{i,aq^{2r_i}}(X') = -1$ or $d_{i,at^{-2}}(X') = -1$.
(b) There exists a monomial X'' appearing in the algorithm such that

$$X'' A_i(zaq^{r_i}t^{-1})^{-1} = X'.$$

a) Firstly we assume (a) is true and X' does contain $Y_i(zaq^{2r_i})^{-1}$ (i.e $d_{i,aq^{2r_i}}(X') = -1$). By assumption, X does not. It implies that the transformation $A_j(zbq^{r_j}t^{-1})^{-1}$ simplifies $Y_i(zaq^{2r_i})^{-1}$ and in particular $i \neq j$. Therefore, the expressions depend on the type of the Dynkin diagram generated by the nodes i and j.

Let us do it in type G_2, $(i,j) = (1,2)$. Thus, $(r_i, r_j) = (3,1)$. Then $bqt^{-1} = aq^6$ and $b = aq^5t$
Let $k > 0$ the maximal integer such that $Y_i(zaq^{6s})^{-1}$ appears in X' for all $0 \leq s \leq k$. We want to prove that the graph representation of the algorithm contains the following subgraph :

$$\begin{array}{cl} X_{k+1} & \\ \downarrow & A_i(zaq^3t^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaq^{6k+3}t^{-1})^{-1} \\ X' & \\ \downarrow & A_j(zaq^6)^{-1} \\ X & \end{array}$$

By definition, $Y_i(zaq^{6(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^{6k}t^{-2})^{-1}$ appears in X', then $Y_i(zaq^{6(k-1)})^{-1}Y_i(zaq^{6k}t^{-2})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zaq^{6k})^{-1}Y_i(zaq^{6(k-1)}t^{-2})^{-1}$ appears in X'.
Thus, X' contains $Y_i(zaq^{6(k-2)})^{-1}Y_i(zaq^{6(k-1)}t^{-2})^{-1}$. We can iterate this reasoning until we get that $Y_i(zat^{-2})^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^{r_j}t^{-1})^{-1} = A_j(zaq^6)^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1 A_i(zaq^{6k+3}t^{-1})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_s A_i(zaq^{6k-6s+9}t^{-1})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zaq^{6u})^{-1}$ appears in X_s for all $0 \leq u \leq k-s$ and $Y_i(zaq^{6(u-1)}t^{-2})^{-1}$ does not appear in X_s by the same argument as above nor $Y_i(zaq^{6(k+1-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1} A_i(zaq^2t^{-1})^{-1} = X_k.$$

Moreover,

$$X_{k+1} A_i(zaq^3t^{-1})^{-1} A_i(zaq^6t^{-1})^{-1} \cdots A_i(zaq^{6k+3}t^{-1})^{-1} = X'. \tag{60}$$

We recall that $X' A_j(zaq^6)^{-1} = X$. By definition of the algorithm, it implies $Y_j(zaq^7t^{-1})$ is admissible in X'. In particular, it implies that $d_{j,aq^7t^{-1}}(X') = 1$ and $d_{j,aq^5t^{-1}}(X') = 0$. But the equation (60) gives

$$d_{j,aq^5t^{-1}}(X') = d_{j,aq^5t^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aq^7t^{-1}}(X') = d_{j,aq^7t^{-1}}(X_{k+1}) + 1.$$

Hence, $d_{j,aq^5t^{-1}}(X_{k+1}) = -1$ and $d_{j,aq^7t^{-1}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaq^5t^{-1})^{-1}$ and does not contain $Y_j(zaq^7t^{-1})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaq^5t^{-1})^{-1} \dots Y_j(zaq^5t^{-2s-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^7t^{-2u-1})^{-1}$, then it contains the monomial $Y_j(zaq^5t^{-2u+1})^{-1}Y_j(zaq^7t^{-2u-1})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^7t^{-2u+1})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaq^7t^{-1})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^6t^{-2(s-u)-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$\begin{gathered} X_{k+s+2}A_j(zaq^6t^{-2})^{-1} \cdots A_j(zaq^6t^{-2s-2})^{-1} \times \\ A_i(zaq^3t^{-1})^{-1} \cdots A_i(zaq^{6k+3}t^{-1})^{-1}A_j(zaq^6)^{-1} = X. \end{gathered} \tag{61}$$

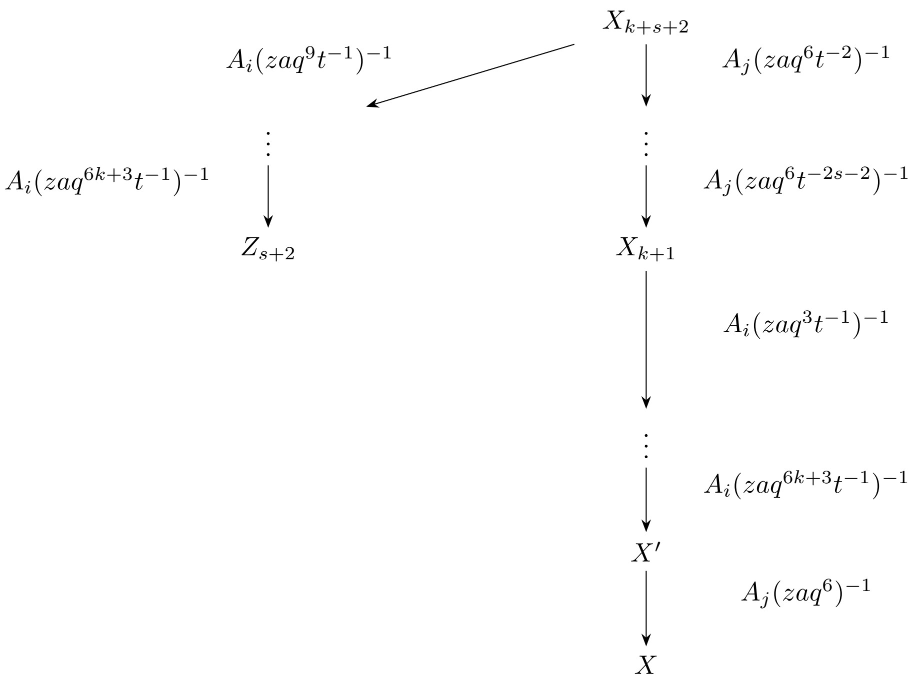
We get the following path in the graph representation of the algorithm:

$$\begin{array}{cl}
X_{k+s+2} & \\
\downarrow & A_j(zaq^6t^{-2})^{-1} \\
\vdots & \\
\downarrow & A_j(zaq^6t^{-2s-2})^{-1} \\
X_{k+1} & \\
\downarrow & A_i(zaq^3t^{-1})^{-1} \\
\vdots & \\
\downarrow & A_i(zaq^{6k+3}t^{-1})^{-1} \\
X' & \\
\downarrow & A_j(zaq^6)^{-1} \\
X &
\end{array}$$

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^{12}t^{-2}) \dots Y_i(zaq^{6k+6}t^{-2})Y_j(zaq^7t^{-3}) \dots Y_j(zaq^7t^{-2s-3}). \tag{62}$$

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaq^3t^{-1})^{-1}$ in order to conclude.
We want to obtain the following subgraph :



To apply the transformation $A_i(zaq^9t^{-1})^{-1}$, we need to check that $Y_i(zaq^{12}t^{-2})$ is admissible in X_{k+s+2}. We know that X_{k+s+2} does not contain $Y_i(zaq^6t^{-2})$ as it would imply that X_{k+s+1} contains $Y_i(zaq^6t^{-2})^2$ which is absurd as all monomials are generic by assumption. Moreover, if X_{k+s+2} contains $Y_i(zaq^{12})$ then the equation (61) implies :

$$d_{i,aq^{12}}(X_{k+s+2}) = d_{i,aq^{12}}(X_k) = 1.$$

But the graph gives that $Y_i(zaq^{12}t^{-2})$ is admissible in X_k. It is absurd. Thus, $Y_i(zaq^{12}t^{-2})$ is admissible in X_{k+s+2} and the algorithm applies the transformation $A_i(zaq^9t^{-1})^{-1}$ to X_{k+s+2}, giving a monomial Z_{k+s+1} such that :

$$X_{k+s+2}A_i(zaq^9t^{-1})^{-1} = Z_{k+s+1}. \tag{63}$$

We assume there exists $1 \leq m \leq k-1$ such that the algorithm gives Z_{u+s+1} for all $m < u \leq k$ such that :

$$\forall m < u \leq k, \quad Z_{u+s+2}A_i(zaq^{6(k-u+1)+3}t^{-1})^{-1} = Z_{u+s+1},$$

setting $Z_{k+s+2} = X_{k+s+2}$. We want to prove that the algorithm gives Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{6(k-m+1)+3}t^{-1})^{-1} = Z_{m+s+1}.$$

We know that

$$Z_{m+s+2} = X_{k+s+2}A_i(zaq^9t^{-1})^{-1} \cdots A_i(zaq^{6(k-m)+3}t^{-1})^{-1}. \tag{64}$$

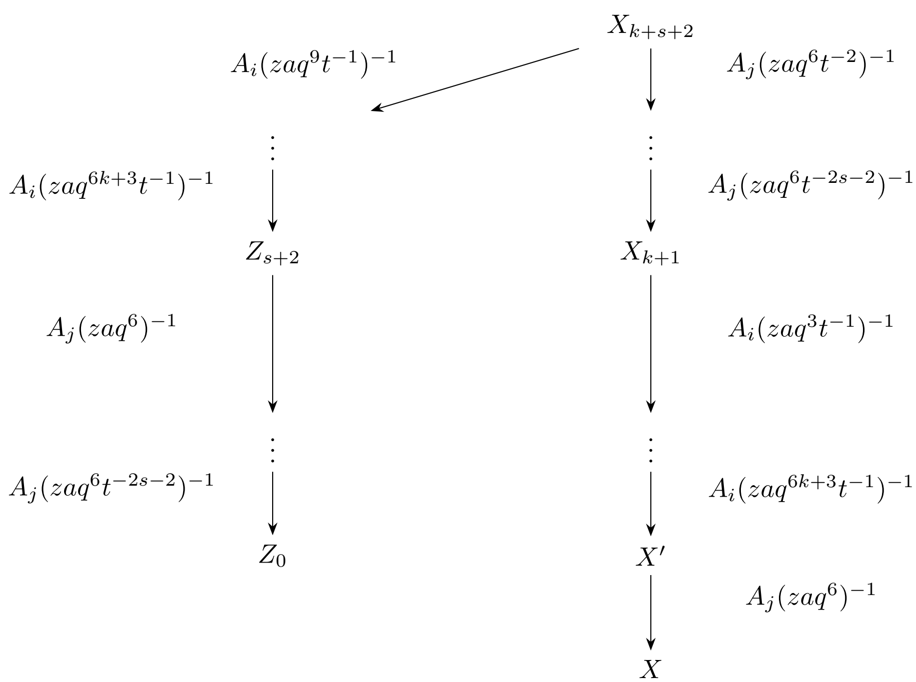
Then, Z_{m+s+2} contains $Y_i(zaq^{6(k-m+2)}t^{-2}) \ldots Y_i(zaq^{6k+6}t^{-2})$.
Let us check that $Y_i(zaq^{6(k-m+2)}t^{-2})$ is admissible in Z_{m+s+2}.
It is clear that Z_{m+s+2} does not contain $Y_i(zaq^{6(k-m+1)}t^{-2})$ as it has been removed at the previous step.

Moreover, if Z_{m+s+2} contains $Y_i(zaq^{6(k-m+2)})$. then by (64), X_{k+s+2} contains $Y_i(zaq^{6(k-m+2)})$. Thus, according to the graph, X_m contains $Y_i(zaq^{6(k-m+2)})$. However, we read in the same figure that $Y_i(zaq^{6(k-m+2)}t^{-2})$ is admissible in X_m. It is absurd. Hence, $Y_i(zaq^{6(k-m+2)}t^{-2})$ is admissible in Z_{m+s+2} and the algorithm gives a monomial Z_{m+s+1} such that :

$$Z_{m+s+2}A_i(zaq^{6(k-m+1)+3}t^{-1})^{-1} = Z_{m+s+1}.$$

By induction, we get monomials $Z_{s+2}, \dots, Z_{k+s+1}$ verifiying the equation (64).

Now we want to complete our graph to the following :



Furthermore, by (63), we get :

$$d_{j,aq^7t^{-1}}(Z_{s+2}) = d_{j,aq^7t^{-1}}(Z_{k+s+1}) = d_{j,aq^7t^{-1}}(X_{k+s+2}) + 1.$$

But we know by admissibility of $Y_j(zaq^7t^{-3})$ and regularity of X_{k+s+2} that :

$$d_{j,aq^7t^{-1}}(X_{k+s+2}) = 0.$$

Hence, $d_{j,aq^7t^{-1}}(Z_{s+2}) = 1$.

Thus, the monomial Z_{s+2} contains $Y_j(zaq^7t^{-1}) \dots Y_j(zaq^7t^{-2s-3})$. We want to check that $Y_j(zaq^7t^{-1})$ is admissible in Z_{s+2}.

If Z_{s+2} contains $Y_j(zaq^5t^{-1})$ then it contains $Y_j(zaq^7t^{-3})Y_j(zaq^5t^{-1})$ (this is of the form $Y_j(zc)Y_j(zcq^{-2r_j}t^2)$). By regularity, it also contains $Y_j(zaq^5t^{-3})$. By (64), X_{k+s+2} contains $Y_j(zaq^5t^{-3})$, and $Y_j(zaq^7t^{-3})$ is not admissible in X_{k+s+2}. It is absurd.

If Z_{s+2} contains $Y_j(zaq^7t)$ then by (64), X_{k+s+2} contains $Y_j(zaq^7t)$. , According to the graph, this implies

that X' contains $Y_j(zaq^7t)$. But $Y_j(zaq^7t^{-1})$ is admissible in X'. It is absurd.
Hence, $Y_j(zaq^7t^{-1})$ is admissible in Z_{s+2} and the algorithm gives a monomial Z_{s+1} such that :

$$Z_{s+2}A_j(zaq^6)^{-1} = Z_{s+1}. \tag{65}$$

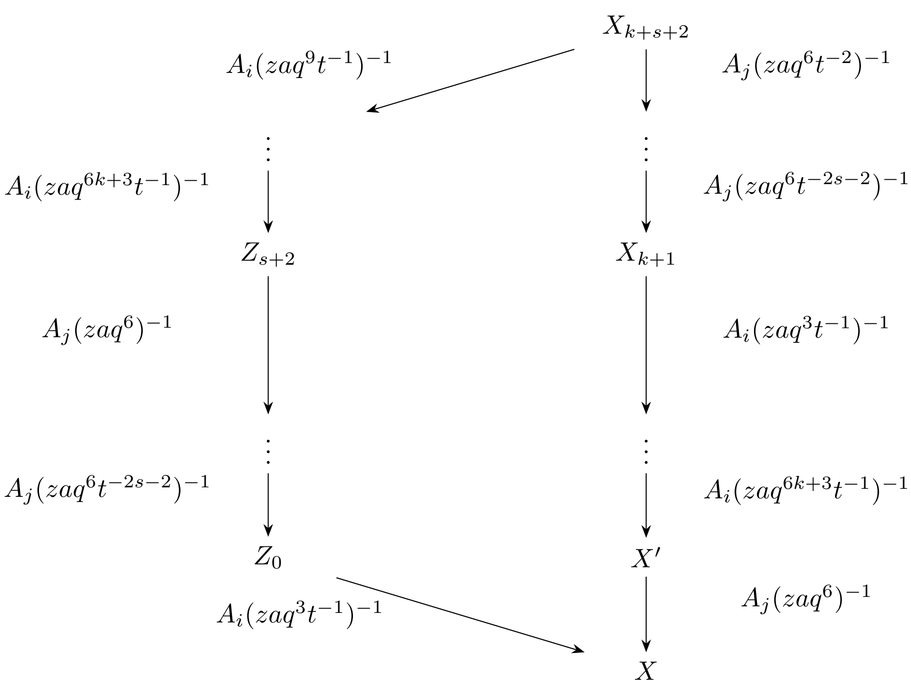
We assume there exists $0 \leq m < s+1$ such that we constructed Z_{u+1} for all $m < u \leq s+1$ such that :

$$\forall m < u \leq s+1, \quad Z_{u+1}A_j(zaq^6t^{-2(s+1-u)})^{-1} = Z_u.$$

We want to prove that $Y_j(zaq^7t^{-2(s-m+1)-1})$ is admissible in Z_{m+1} so that the algorithm gives the right monomial Z_m.
It is clear that Z_{m+1} does not contain $Y_j(zaq^7t^{-2(s-m+1)+1})$ as it has been removed at the previous step.
Moreover, if Z_{m+1} contains $Y_j(zaq^5t^{-2(s-m+1)-1})$ then by construction, X_{k+s+2} also contains $Y_j(zaq^5t^{-2(s-m+1)-1})$.
According to the graph, this implies that X_{k+m+2} contains $Y_j(zaq^5t^{-2(s-m+1)-1})$. However, we read in the graph that $Y_j(zaq^5t^{-2(s-m+1)-1})$ is admissible in X_{k+m+2}. It is absurd. Hence, $Y_j(zaq^7t^{-2(s-m+1)-1})$ is admissible in Z_{m+1} and the algorithm gives a monomial Z_m such that :

$$Z_{m+1}A_j(zaq^6t^{-2(s-m+1)})^{-1} = Z_m.$$

Finally, let us prove that $Y_i(zaq^6t^{-2})$ is admissible in Z_0.
We have the equality $Z_0A_i(zq^3t^{-1})^{-1} = X$. Then, by identification of the degrees the admissibility is clear.
Hence the following graph :



Now, we assume $A_j(zbq^{r_j}t^{-1})$ simplifies $Y_i(zat^2)$.

We have $d_{i,at^{-2}}(X') = -1$. We know that $d_{i,at^{-2}}(X) = 0$, then it implies

$$d_{i,at^{-2}}(A_j(zbqt^{-1})^{-1}) = 1.$$

Hence, $bqt^{-1} = at^{-2}$ and hence

$$b = aq^{-1}t^{-1}.$$

As in the previous case, we want to apply the induction hypothesis and get the following graph for a k defined above :

$$\begin{array}{cl} X_{k+1} & \\ \downarrow & A_i(zaq^3t^{-1})^{-1} \\ \vdots & \\ \downarrow & A_i(zaq^3t^{-2k-1})^{-1} \\ X' & \\ \downarrow & A_j(zat^{-2})^{-1} \\ X & \end{array}$$

Let $k > 0$ the maximal integer such that $Y_i(zat^{-2s})^{-1}$ appears in X' for all $0 \leq s \leq k$.
By definition, $Y_i(zat^{-2(k+1)})^{-1}$ is not in X'. Moreover, if $Y_i(zaq^6t^{-2k})^{-1}$ appears in X', then $Y_i(zat^{-2(k-1)})^{-1}Y_i(zaq^6t^{-2k})^{-1}$ also appears in X'. The monomial has to be regular, so $Y_i(zat^{-2k})^{-1}Y_i(zaq^6t^{-2(k-1)})^{-1}$ appears in X'.
Thus, X' contains $Y_i(zat^{-2(k-2)})^{-1}Y_i(zaq^6t^{-2(k-1)})^{-1}$. We can iterate this reasoning until we get that $Y_i(zaq^6)^{-1}$ appears in X'. However it does not appear in X and has to be simplified by the transformation $A_j(zbq^{r_j}t^{-1})^{-1} = A_j(zat^{-2})^{-1}$. It is absurd by definition of A_j. Hence, we can apply the induction hypothesis and we obtain that there exists X_1 given by the algorithm such that

$$X_1A_i(zaq^3t^{-(2k+1)})^{-1} = X'.$$

We can define recursively X_s, $2 \leq s \leq k+1$ such that

$$X_sA_i(zaq^3t^{-(2k-2s+3)})^{-1} = X_{s-1}.$$

Indeed, for all $1 \leq s \leq k+1$, $Y_i(zat^{-2u})^{-1}$ appears in X_s for all $0 \leq u \leq k-s$ and $Y_i(zaq^6t^{-2(u-1)})^{-1}$ does not appear in X_s by the same argument as above, nor $Y_i(zat^{-2(k+2-s)})^{-1}$ so that we can apply the induction hypothesis. In particular, we have constructed X_{k+1} such that

$$X_{k+1}A_i(zaq^3t^{-1})^{-1} = X_k.$$

Moreover,

$$X_{k+1}A_i(zaq^3t^{-1})^{-1}A_i(zaq^3t^{-3})^{-1}\cdots A_i(zaq^3t^{-(2k+1)})^{-1} = X'. \tag{66}$$

By definition of the algorithm, the arrow $X' \to X$ implies that $Y_j(zaqt^{-3})$ is admissible in X'. In particular, it implies that $d_{j,aqt^{-3}}(X') = 1$ and $d_{j,aqt^{-1}}(X') = 0$. But the equation (66) gives

$$d_{j,aqt^{-1}}(X') = d_{j,aqt^{-1}}(X_{k+1}) + 1 \quad \text{and} \quad d_{j,aqt^{-3}}(X') = d_{j,aqt^{-3}}(X_{k+1}) + 1.$$

Hence, $d_{j,aqt^{-1}}(X_{k+1}) = -1$ and $d_{j,aqt^{-3}}(X_{k+1}) = 0$. Thus, X_{k+1} contains $Y_j(zaqt^{-1})^{-1}$ and does not contain $Y_j(zaqt^{-3})^{-1}$.

Let $s \geq 0$ be the maximal integer such that X_{k+1} contains the monomial :

$$Y_j(zaqt^{-1})^{-1} \dots Y_j(zaq^{2s+1}t^{-1})^{-1}.$$

If for $1 \leq u \leq s$, X_{k+1} contains $Y_j(zaq^{u+1}t^{-3})^{-1}$, then it contains the monomial $Y_j(zaq^{2(u-1)+1}t^{-1})^{-1}Y_j(zaq^{2u+1}t^{-3})^{-1}$. However, X_{k+1} has to be regular, so X_{k+1} also contains $Y_j(zaq^{2(u-1)+1}t^{-3})^{-1}$. Iterating the same argument we would get that X_{k+1} does contain $Y_j(zaqt^{-3})^{-1}$, which is absurd by the argument written below. Hence, we can use $s+1$ times the recurrence hypothesis until a monomial X_{k+s+2}. We get :

$$\forall 0 \leq u \leq s, \quad X_{k+u+2}A_j(zaq^{2(s-u)+2}t^{-2})^{-1} = X_{k+u+1}$$

At this step, we have constructed a path from X_{k+s+2} to X with:

$$X_{k+s+2}A_j(zaq^2t^{-2})^{-1} \cdots A_j(zaq^{2s+2}t^{-2})^{-1} \times$$

$$A_i(zaq^3t^{-1})^{-1} \cdots A_i(zaq^3t^{-2k-1})^{-1}A_j(zat^{-2})^{-1} = X.$$

We get the following path in the graph representation of the algorithm:

$$\begin{array}{cl}
X_{k+s+2} & \\
\downarrow & A_j(zaq^2t^{-2})^{-1} \\
\vdots & \\
\downarrow & A_j(zaq^{2s+2}t^{-2})^{-1} \\
X_{k+1} & \\
\downarrow & A_i(zaq^3t^{-1})^{-1} \\
\vdots & \\
\downarrow & A_i(zaq^3t^{-2k-1})^{-1} \\
X' & \\
\downarrow & A_j(zat^{-2})^{-1} \\
X &
\end{array}$$

Finally,

$$X_{k+s+2} \text{ contains } Y_i(zaq^6t^{-4}) \dots Y_i(zaq^6t^{-2k-2})Y_j(zaq^3t^{-3}) \dots Y_j(zaq^{2s+3}t^{-3}). \tag{67}$$

Let us prove that $s > 1$. By contradiction, if $s \leq 1$ then $Y_i(zaq^6t^{-2})$ is admissible in X_{k+s+2}. Indeed,

$$d_{i,aq^6t^{-2}}(X_{k+s+2}) = d_{i,aq^6t^{-2}}(X_{k+1}) = 1,$$

$$d_{i,aq^6}(X_{k+s+2}) = d_{i,aq^6}(X_{k+1}) = 0,$$

$$d_{i,at^{-2}}(X_{k+s+2}) = d_{i,at^{-2}}(X_{k+1}) = 0.$$

Hence, the algorithm applies the transformation $A_i(zaq^3t^{-1})^{-1}$ to X_{k+s+2} and we get a monomial X^* such that :

$$X_{k+s+2}A_i(zaq^3t^{-1})^{-1} = X^*.$$

Then in particular,

$$d_{j,aqt^{-1}}(X^*) = d_{j,aqt^{-1}}(X_{k+s+2}) + 1 = d_{j,aqt^{-1}}(X_{k+s+1}) + 1 + 1 = 1.$$

Indeed, by genericity, $d_{j,aqt^{-1}}(X_{k+s+1})$ has to be equal to -1. Moreover, by admissibility of $Y_j(zaq^3t^{-3})$ in X_{k+s+2} and regularity,

$$d_{j,aq^3t^{-3}}(X^*) = d_{j,aq^3t^{-3}}(X_{k+s+2}) = 1,$$

$$d_{j,aqt^{-3}}(X^*) = d_{j,aqt^{-3}}(X_{k+s+2}) = 0,$$

Then X^* contains a term of the form $Y_j(zc)Y_j(zcq^{-2r_j}t^2)$ but does not contain $Y_j(zcq^{-2r_j})$ (with $c = q^3t^{-3}$) and X^* is not regular. It is absurd. Hence, $s > 1$.

The aim is now to construct another path from X_{k+s+2} to X ending with the transformation $A_i(zaq^3t^{-1})^{-1}$ in order to conclude.
Firstly,

$$d_{i,aq^6t^{-4}}(X_{k+s+2}) = d_{i,aq^6t^{-4}}(X_k) = 1,$$

because as shown in the graph below, $Y_i(zaq^6t^{-4})$ is admissible in X_k. Moreover, $d_{i,at^{-4}}(X_{k+s+2}) = d_{i,at^{-4}}(X_k) = 0$ for the same argument of admissibility. Additionally, if $d_{i,aq^6t^{-2}}(X_{k+s+2}) = 1$ then by definition of $A_j(zaq^4t^{-2})^{-1}$ it implies $d_{i,aq^6t^{-2}}(X_{k+s}) = 2$ which is impossible by the genericity hypothesis.
Hence, $Y_i(zaq^6t^{-4})$ is admissible in X_{k+s+2}.

We construct

$$Z_{k+s+1} := X_{k+s+2}A_i(zaq^3t^{-3})^{-1}.$$

We assume there exists $1 \le m \le k$ such that we constructed Z_{s+u+1}, for $m < u \le k$ such that

$$Z_{u+s+1} = Z_{u+s+2}A_i(zaq^3t^{-2(k-u)-3})^{-1}.$$

Let us construct Z_{s+m+1}.
Firstly, according to the graph,

$$d_{i,aq^6t^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,aq^6t^{-2(k-m)-4}}(X_m) = 1.$$

Moreover,

$$d_{i,aq^6t^{-2(k-m)-2}}(Z_{s+m+2}) = d_{i,aq^6t^{-2(k-m)-2}}(Z_{s+m+3}) - 1 = d_{i,aq^6t^{-2(k-m)-2}}(X_{m+1}) - 1 = 0,$$

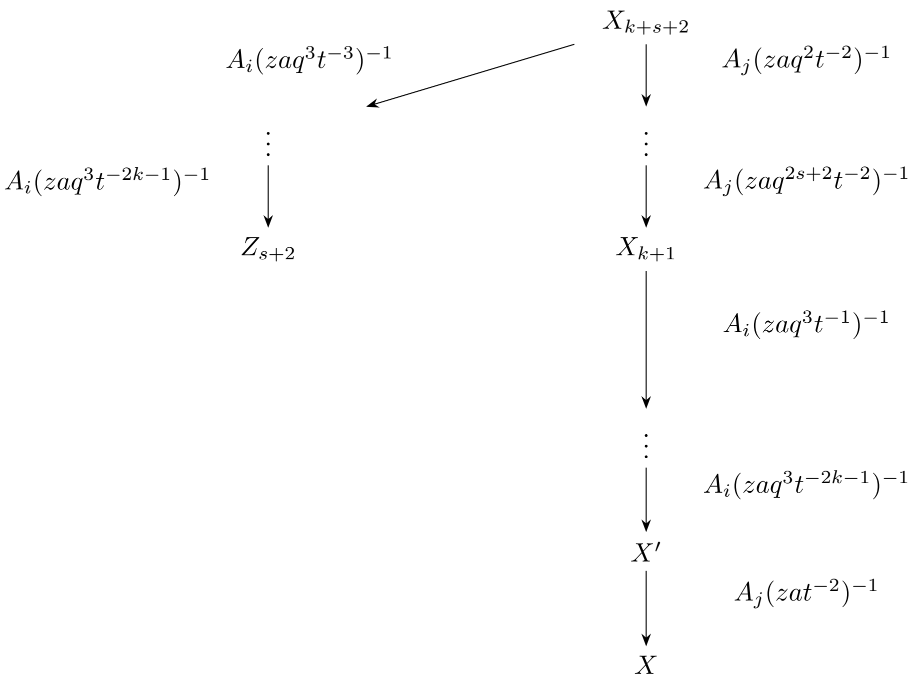
because of the transformation $Z_{s+m+3} \to Z_{s+m+2}$ and because $Y_i(zaq^6t^{-2(k-m)-2})$ is admissible in X_{m+1}. Additionally, $Y_i(zaq^6t^{-2(k-m)-4})$ is admissible in X_m. Thus,

$$d_{i,at^{-2(k-m)-4}}(Z_{s+m+2}) = d_{i,at^{-2(k-m)-4}}(X_{k+s+2}) = d_{i,at^{-2(k-m)-4}}(X_m) = 0.$$

Hence the admissibility of $Y_i(zaq^6t^{-2(k-m)-4})$ in Z_{s+m+2} and the existence of the following transformation given by the algorithm :

$$Z_{m+s+1} = Z_{m+s+2}A_i(zaq^3t^{-2(k-m)-3})^{-1}.$$

We obtain the following graph :



Again, we want to construct a path from Z_{s+2} to X in the algorithm graph.
Firstly, by definition of the variables A_i and by admissibility in X',

$$\begin{aligned} d_{j,aqt^{-3}}(Z_{s+2}) &= d_{j,aqt^{-3}}(X_{k+s+2}) + 1, \\ &= d_{j,aqt^{-3}}(X_k) + 1, \\ &= d_{j,aqt^{-3}}(X_{k-1}) - 1 + 1, \\ &= d_{j,aqt^{-3}}(X') = 1. \end{aligned}$$

Moreover, the graph gives :

$$d_{j,aq^{-1}t^{-3}}(Z_{s+2}) = d_{j,aq^{-3}t^{-3}}(X') = 0,$$

as $Y_j(zaqt^{-3})$ is admissible in X'. Additionally, we deduce from the graph the following computation

$$\begin{aligned} d_{j,aqt^{-1}}(Z_{s+2}) &= d_{j,aqt^{-1}}(X_{k+s+2}) \\ &= d_{j,aqt^{-1}}(X_{k+s+1}) - 1 \\ &= d_{j,aqt^{-1}}(X_k) + 1 - 1 \\ &= d_{j,aqt^{-1}}(X') = 0 \end{aligned}$$

Hence, $Y_j(zaqt^{-3})$ is admissible in Z_{s+2}. Thus, the algorithm constructs the following monomial :

$$Z_{s+1} = Z_{s+2}A_j(zat^{-2})^{-1}.$$

Now, we assume there exists $0 \leq m \leq s$ such that we constructed Z_{u+1}, for $m \leq u \leq s$ such that

$$Z_{u+1} = Z_{u+2}A_j(zaq^{2(s-u)}t^{-2})^{-1}.$$

Let us construct Z_m.
Let us prove that $Y_j(zaq^{2(s+1-m)+1}t^{-3})$ is admissible in Z_{m+1}.
Firstly, according to the graph and by admissibility,

$$d_{j,aq^{2(s+1-m)+1}t^{-3}}(Z_{m+1}) = d_{j,aq^{2(s+1-m)}t^{-3}}(X_{k+m+2}) = 1.$$

Moreover,

$$d_{j,aq^{2(s-m)+1}t^{-3}}(Z_{m+1}) = d_{j,aq^{2(s-m)+1}t^{-3}}(Z_{m+2}) - 1 = 0,$$

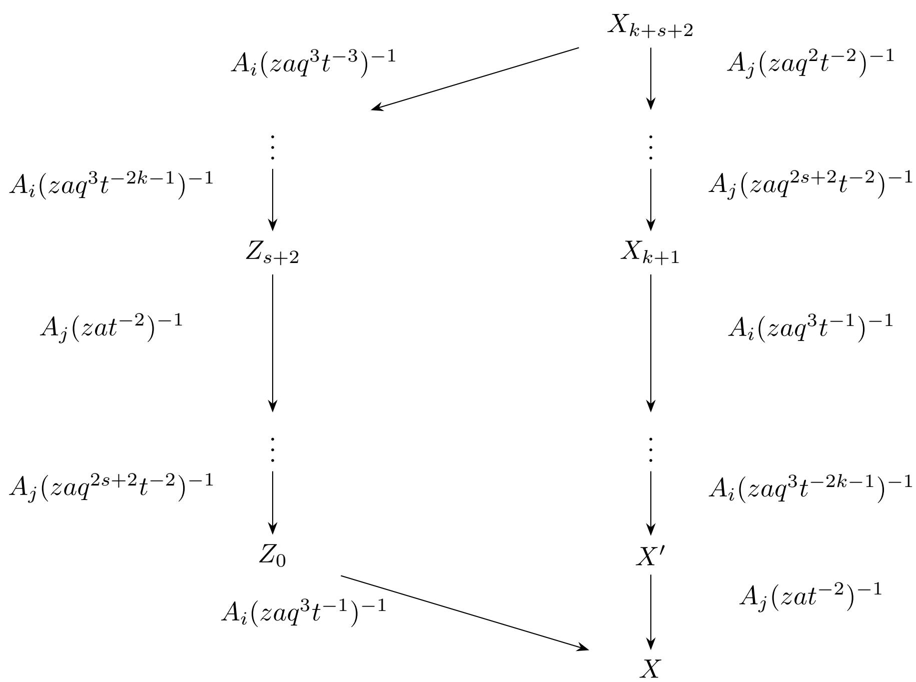
by admissibility and because of the transformation $Z_{m+2} \to Z_{m+1}$.
Additionally, $Y_j(zaq^{2(s+1-m)+1}t^{-3})$ is admissible in X_{k+m+2}. Thus,

$$d_{j,aq^{2(s+1-m)+1}t^{-1}}(Z_{m+1}) = d_{j,aq^{2(s+1-m)+1}t^{-1}}(X_{k+m+2}) = 0.$$

Hence the admissibility of $Y_j(zaq^{2(s+1-m)+1}t^{-3})$ in Z_{m+1} and the existence of the following transformation given by the algorithm :

$$Z_m = Z_{m+1}A_j(zaq^{2(s+1-m)}t^{-2})^{-1}.$$

To conclude, we have to prove that $Y_i(zaq^6t^{-2})$ is admissible in Z_0.
We have the equality $Z_0A_i(zq^3t^{-1})^{-1} = X$. Then, by identification of the degrees the admissibility is clear.
Hence the following paths of transformations in the algorithm :



b) We assume there exists a monomial X'' also appearing in the algorithm such that

$$X''A_i(zaq^{r_i}t^{-1})^{-1} = X',$$

then X'' does contain $Y_i(zaq^{2r_i}t^{-2})$.

Here there are two cases to consider :

- $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_i(zaq^{2r_i}t^{-2})$ comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$.
- X'' contains $Y_j(zaq^{2r_j}t^{-2})$.

Firstly, we assume $d_{j,bq^{2r_j}t^{-2}}(A_i(zaq^{r_i}t^{-1})^{-1}) = 1$ and the term $Y_j(zaq^{2r_j}t^{-2})$ in X' comes from the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$. Hence, $i \neq j$, and we assume the Dynkin subdiagram with nodes $\{i,j\}$ is of type G_2 with $(i,j) = (1,2)$.
Thus,

$$bq^2t^{-2} = aqt^{-1} \qquad \text{or} \qquad bq^2t^{-2} = aq^3t^{-1} \qquad \text{or} \qquad bq^2t^{-2} = aq^5t^{-1}$$

If $bq^2t^{-2} = aqt^{-1}$ then $a = bqt^{-1}$ and $d_{i,a}(A_j(zbqt^{-1})^{-1}) = 1$. However, $d_{i,a}(X) = -1$ and $X'A_j(zbq^{r_j}t^{-1})^{-1} = X$. Thus, $d_{i,a}(X') = -2$ which is absurd because it contradicts the genericity.
Hence, $bq^2t^{-2} = aq^3t^{-1}$ or $bq^2t^{-2} = aq^5t^{-1}$. By admissibility, $d_{j,bq^2t^{-2}}(X') = 1$ and $d_{j,bt^{-2}}(X') = 0$.
But here, $d_{j,bt^{-2}}(A_i(zaq^3t^{-1})^{-1}) = 1$. Thus, $d_{j,bt^{-2}}(X'') = -1$.
Let $s \geq 0$ be the maximal integer such that X'' contains the monomial :

$$Y_j(zbt^{-2})^{-1} \dots Y_j(zbt^{-2s-2})^{-1}.$$

If for one $0 \leq u \leq s$, X'' contains $Y_j(zbq^2t^{-2u-2})^{-1}$ by regularity we deduce that X'' contains $Y_j(zbq^2t^{-2})^{-1}$.
But we have :

$$d_{j,bq^2t^{-2}}(X'') = d_{j,bq^2t^{-2}}(X') - d_{j,bq^2t^{-2}}(A_i(zaq^3t^{-1})^{-1}) = 1 - 1 = 0.$$

Henceit is absurd. Thus, the induction hypoyhesis allow to construct $X_1, \dots, X_{s+1}$ such that we get the following subgraph in the algorithm graph :

$$\begin{array}{cl}
X_{s+1} & A_j(zbqt^{-3})^{-1} \\
\downarrow & \\
\vdots & \\
\downarrow & \\
X_1 & A_j(zbqt^{2s-3})^{-1} \\
\downarrow & \\
X'' & A_i(zaq^3t^{-1})^{-1} \\
\downarrow & \\
X' & A_j(zbqt^{-1})^{-1} \\
\downarrow & \\
X &
\end{array}$$

In particular, by admissibility and regularity,

$$d_{j,bq^2t^{-4}}(X_{s+1}) = 1, \qquad d_{j,bt^{-2}} = 0, \qquad d_{j,bt^{-4}} = 0.$$

Now, $d_{i,c}(X_{s+1}) = d_{i,c}(X'')$ for $c = aq^4t^{-2}, aq^4, at^{-2}$ Hence, the admissibility of $Y_i(zaq^6t^{-2})$ in X'' implies its admissibility in X_{s+1}, and the creation of

$$Z = X_{s+1}A_i(zaq^3t^{-1})^{-1}$$

in the algorithm. Moreover $d_{j,bt^{-2}}(A_i(zaq^3t^{-1})^{-1}) = 1$, and $d_{j,bt^{-4}}(A_i(zaq^3t^{-1})^{-1}) = 0$. Hence, $d_{j,bq^2t^{-4}}(Z) = d_{j,bt^{-2}}(Z) = 1$. and $d_{j,bt^{-4}}(Z) = 0$. This contradicts the regularity of Z.

Hence, we are in the second case and X'' contains $Y_j(zaq^{2r_j}t^{-2})$.
Let us prove that $Y_j(zbq^{2r_j}t^{(}-2))$ is admissible in X''.
If $i = j$ (so that $r_i = r_j$), then by contradiction we assume that $Y_j(zbq^{2r_j}t^{-2})$ is not admissible in X''. We know that $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X' so that the presence of $Y_i(zaq^{2r_i}t^{-2})$ in X'' prevents from doing the transformation $A_j(bq^{r_j}t^{-1})^{-1}$. It implies that $a = bq^{-2r_i}$ or $a = bt^2$. This implies X contains $Y_i(zb)^{-1} = Y_i(zaq^{2r_i})^{-1}$ or $Y_i(zat^{-2})^{-1}$. It is absurd by the initial assumption on X. Thus, $Y_j(zbq^{2r_j}t^{-2})$ is admissible in X''.
If $i \neq j$ then it is clear that the admissibility of $Y_j(zbq^{2r_j}t^{(}-2))$ in X' implies its admissibility in X''.
Finally, there exists a monomial

$$X_{new} = X''A_j(zaq^{r_j}t^{-1})^{-1}$$

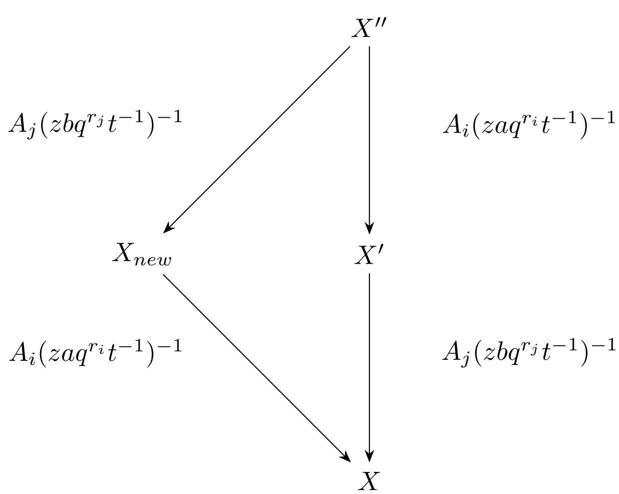
given by the algorithm and X_{new} contains $Y_i(zaq^{2r_i}t^{-2})$. Moreover, if the transformation $A_i(zaq^{r_i}t^{-1})^{-1}$ is not admissible from X_{new} then there exists a factor $Y_i(zat^{-2})$ (resp. $Y_i(zaq^{2r_i})$) blocking this transformation. But this would imply that this factor also appears in X as we have the following equality of monomials :

$$X_{new} = A_i(zaq^{r_i}t^{-1})^{-1}X.$$

We remark that a priori, this equality is not sufficient to get an arrow from X_{new} to X. This equality implies :

$$d_{i,aq^{2r_i}}(X_{new}) = d_{i,aq^{2r_i}}(X),$$

and same for at^{-2}. And hence X does contain $Y_i(zc)$ and $Y_i(zct^2)^{-1}$ with $c = at^{-2}$ (resp. $Y_i(zcq^{-2r_i})^{-1}$ with $c = aq^{2r_i}$) which contradicts the regularity of X. It is impossible. Hence the admissibility of $Y_i(zaq^{2r_i}t^{-2})$ in X_{new}.
Finally we get the following paths :



This concludes the proof in type G_2. □

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