

# Research Statement

## 1. INTRODUCTION

My research lies in representation theory. I work on quantum groups, vertex algebras and their links with mathematical physics. My work focuses on *deformed  $\mathcal{W}$ -algebras*  $\mathcal{W}_{q,t}(\mathfrak{g})$  defined by Frenkel and Reshetikhin [FR2]. It is a family of associative algebras with two parameters  $q$  and  $t$ , attached to a simple Lie algebra  $\mathfrak{g}$ . These algebras are generated by coefficients of formal series commuting with screening operators.

These objects appear in various areas of mathematics and mathematical physics: they are deformations of a particular family of graded vertex algebras called  $\mathcal{W}$ -algebras.  $\mathcal{W}$ -algebras are defined by quantized Drinfeld-Sokolov reduction [FF1]; they appeared in the study of the classification of two-dimensional rational conformal field theories [FL1, FL2, Z]. Their deformations  $\mathcal{W}_{q,t}(\mathfrak{g})$ , defined by Frenkel and Reshetikhin [FR2], are then linked with the Feigin-Frenkel duality. The deformed  $\mathcal{W}$ -algebras are also strongly related to  $qq$ -character theory and have thus found applications in the study of four-dimensional supersymmetric gauge theories [Ne, KP1, AFO, KP2, EP, FJMV, FJM]. Furthermore, many observations suggest that they interpolate the representation rings of the quantum affine algebras  $U_q(\widehat{\mathfrak{g}})$  and  $U_t({}^L\widehat{\mathfrak{g}})$ . These are Hopf algebras defined by Drinfeld and Jimbo, which are linked to XXZ-type quantum integrable models and Bethe ansatz equations [OW, RW, R]. Finally, the study of various specializations of the deformed  $\mathcal{W}$ -algebras  $\mathcal{W}_{q,t}(\mathfrak{g})$  is possibly linked with the geometric Langlands correspondence [FH1].

In Section 2, I recall the background. In Section 3, I present a previous work and my recent paper [A2] in which I define a formal setting for  $\mathcal{W}_{q,t}(\mathfrak{g})$ , I give an algorithm which constructs elements of  $\mathcal{W}_{q,t}(\mathfrak{g})$ , and I use it to prove a conjecture of Frenkel and Reshetikhin in new cases. In Section 4, I describe current work on the range of this algorithm. In Section 5, I describe future projects.

## 2. BACKGROUND

Let  $\mathfrak{g}$  be a simple Lie algebra of rank  $\ell$ , with Cartan matrix  $C = (C_{ij})_{i,j \in I}$ , where  $I = \{1, \dots, \ell\}$ . Let  $r^\vee$  be its lacing number and  $r_i = r^\vee(\alpha_i, \alpha_i)/2$ , where long roots have square length 2. The Langlands dual Lie algebra  ${}^L\mathfrak{g}$  has Cartan matrix  ${}^tC$ .

**2.1.  $\mathcal{W}$ -algebras and quantum affine algebras.** For  $k \in \mathbb{C}$ , the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g})$  is the quantum Drinfeld-Sokolov reduction of the affine vertex algebra  $V^k(\mathfrak{g})$  [FF1]. For generic  $k$ , it is the intersection of the kernels of screening operators on a Heisenberg vertex algebra. Two facts are important for my work.

- **Feigin-Frenkel duality.** We have  $\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}^{k'}({}^L\mathfrak{g})$  when  $r^\vee(k + h^\vee)(k' + {}^Lh^\vee) = 1$  [FF2]. In the free field realization there are two families of screening operators, and the duality exchanges them.
- **Critical level.** At  $k = -h^\vee$ , the vertex algebra  $V^{-h^\vee}(\mathfrak{g})$  has a large center  $\mathfrak{z}(\widehat{\mathfrak{g}})$ . Feigin and Frenkel proved that  $\mathfrak{z}(\widehat{\mathfrak{g}})$  is isomorphic to the classical  $\mathcal{W}$ -algebra of  ${}^L\mathfrak{g}$ , that is, to  $\text{Fun Op}_{{}^L G}(D)$ , the functions on  ${}^L G$ -opers on the disc [FF3, Fre2]. Here the Langlands dual group appears on its own, out of a vertex algebra attached to  $\mathfrak{g}$  alone.

The Langlands dual Lie algebra appears in both results, and the reason for this is not obvious. One motivation to study quantum deformations of these objects is to understand it better [Fre1].

Around 1985, Drinfeld and Jimbo defined the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  as a quantization of the universal enveloping algebra of the affine Kac-Moody algebra  $\widehat{\mathfrak{g}}$ . For a generic  $q \in \mathbb{C}^\times$ , the simple finite-dimensional modules of  $U_q(\widehat{\mathfrak{g}})$  are classified by Drinfeld polynomials [CP]; for  $i \in I$ , we write  $V_{i,a}$  for the fundamental module of highest  $\ell$ -weight  $Y_{i,a}$ . Frenkel and Reshetikhin introduced the  $q$ -character map [FR3]

$$\chi_q : K_0(\text{Rep } U_q(\widehat{\mathfrak{g}})) \longrightarrow \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times},$$

an injective ring homomorphism whose image is generated by the  $q$ -characters of the fundamental modules. Frenkel and Mukhin proved that this image is the intersection of the kernels of screening operators  $S_i$  on  $\mathcal{Y}$  [FM]. A module is *special* if its  $q$ -character has only one dominant monomial,

and *thin* if all coefficients in its  $q$ -character equal 1. For special modules, the Frenkel-Mukhin algorithm computes the  $q$ -character: it starts from the highest monomial and multiplies step by step by root-type variables  $A_{i,a}^{-1}$  [FM].

This construction carries an application in quantum integrable systems. From the universal  $R$ -matrix, each finite-dimensional module  $V$  gives a transfer-matrix  $t_V(z) = \sum t_V[n]z^n$ , and all  $t_V[n]$ ,  $n \in \mathbb{Z}$  commute with each other. They are the Hamiltonians of a quantum integrable system, and the main problem is to describe their eigenvalues. In [FR3] Frenkel and Reshetikhin conjecture that if in  $\chi_q(V)$  we replace each  $Y_{i,a}$  by a ratio  $Q_i(azq^{-r_i})/Q_i(azq^{r_i})$ , where the  $Q_i$  are polynomials, we get the eigenvalues of  $t_V(z)$ . Frenkel and Hernandez proved this conjecture in [FH3].

**2.2. Deformed  $\mathcal{W}$ -algebras.** Deformed  $\mathcal{W}$ -algebras were first defined for  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_N$  [SKAO, AKOS, FF4], and then for every simple Lie algebra [FR2]. Frenkel and Reshetikhin define the deformed Cartan matrix

$$C_{ij}(q, t) = (q^{r_i}t^{-1} + q^{-r_i}t)\delta_{ij} - [I_{ij}]_q, \quad I = 2\text{Id} - C,$$

and  $B(q, t) = D(q, t)C(q, t)$  with  $D(q, t) = \text{diag}([r_1]_q, \dots, [r_\ell]_q)$ . The Heisenberg algebra  $\mathcal{H}_{q,t}(\mathfrak{g})$  has generators  $a_i[n]$  and relations

$$[a_i[n], a_j[m]] = \frac{1}{n} (q^n - q^{-n})(t^n - t^{-n}) B_{ij}(q^n, t^n) \delta_{n,-m}, \quad n, m \in \mathbb{Z} \setminus \{0\}$$

From these generators they build currents  $Y_i(z)$  (of fundamental weight type),  $A_i(z)$  (of simple root type), and two families of screening currents  $S_i^\pm(z)$ . The screening currents satisfy the difference equations

$$S_i^+(zq^{-r_i}) = t^{-2}q^{2r_i} : A_i(z) S_i^+(zq^{r_i}) :, \quad S_i^-(zt) = t^{-2}q^{2r_i} : A_i(z) S_i^-(zt^{-1}) :.$$

The *deformed chiral algebra*  $\mathbf{H}_{q,t}(\mathfrak{g})$  denotes the vector space spanned by the monomials in the  $Y_i(za)^{\pm 1}$  and their derivatives, and  $\mathbf{W}_{q,t}(\mathfrak{g})$  denotes its subspace commuting with the residues  $\text{Res } S_i^\pm$ . The algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$  denotes the algebra generated by the Fourier coefficients of the fields in  $\mathbf{W}_{q,t}(\mathfrak{g})$ . Throughout, bold letters denote spaces of fields, and calligraphic letters the algebras generated by their Fourier coefficients.

In [FR2], Frenkel and Reshetikhin highlight a curious link between those fields and the Grothendieck ring of the finite dimensional representations of  $U_q(\widehat{\mathfrak{g}})$ : the limit  $t \rightarrow 1$  appears to be a linear combination of  $q$ -characters. In particular, they formulate the following conjecture:

**Conjecture 1** ([FR2, Conjecture 1]). *For each  $i \in I$ , there is a field  $T_i(z)$  which commutes with all screening operators, of the form*

$$T_i(z) = Y_i(z) + \sum c(q, t) : Y_i(z) A_{i_1}(zq^{a_1}t^{b_1})^{-1} \dots A_{i_k}(zq^{a_k}t^{b_k})^{-1} :,$$

where each  $c(q, 1)$  is a positive integer which does not depend on  $q$ , and such that the weights of these terms, counted with multiplicity  $c(q, 1)$ , are the weights of  $V_{i,a}$ , where the weight of a monomial  $:\prod_j Y_{i_j}(za_j)^{\epsilon_j}:$  is  $\sum_j \epsilon_j \omega_{i_j}$ .

This conjecture was known in type  $A_\ell$  [FF4, AKOS], for  $i = 1$  in classical types [FR2], and in rank 2 [BP]. The elements of  $\mathbf{W}_{q,t}(\mathfrak{g})$  are closely related to Nekrasov's  $qq$ -characters [Ne, KP1], which were studied with combinatorial methods by Feigin, Jimbo and Mukhin [FJMV, FJM]. The combinatorics have a lot in common, and the underlying reason is not understood.

### 3. RECENT WORK

**3.1. An earlier project: Yang–Baxter characters of wreath products.** In a separate project [A1], I studied the characters of infinite discrete groups which come from the Yang–Baxter equation. A *character* of a group  $G$  is a central positive-definite function normalized at the unit; the extremal ones generalize the irreducible characters for finite dimensional representations. For the infinite symmetric group  $\mathfrak{S}_\infty$ , Thoma classified them by a pair of decreasing sequences  $(\alpha, \beta)$  with  $\sum_i (\alpha_i + \beta_i) \leq 1$ . An involutive solution  $R \in \text{End}(V \otimes V)$  of the Yang–Baxter equation gives a representation of  $\mathfrak{S}_\infty$  on  $\bigotimes_{n \geq 1} \text{End}(V)$ , sending  $\sigma_i$  to  $R$  acting in the positions  $i, i+1$ , and composing with the normalized trace gives an extremal character. Lechner, Pennig and Wood determined which Thoma parameters arise in this way [LPW]: those with finitely many non-zero terms, with rational numbers, and with  $\sum_i (\alpha_i + \beta_i) = 1$ .

In [A1] I extend this picture to the wreath product  $G = T \wr \mathfrak{S}_\infty$ , where  $T$  is a finite group. From a representation  $\pi$  of  $T$  and an involutive  $R$ -matrix  $R$  I build a canonical candidate representation of  $G$ , and I prove that it is a representation if and only if  $R$  satisfies the Yang–Baxter equation and the pair  $(\pi, R)$  satisfies the *extended reflection equation*

$$R_1 \pi(t) R_1 \pi(t') = \pi(t') R_1 \pi(t) R_1, \quad t, t' \in T,$$

which I read off from a braid-diagram calculus. Hirai and Hirai classified the extremal characters of  $G$  by families of parameters  $(a_{\zeta, \varepsilon})_{\zeta \in \hat{T}, \varepsilon \in \{0,1\}}$  and  $(\mu_\zeta)_{\zeta \in \hat{T}}$ , indexed by the irreducible characters of  $T$  [HH].

**Theorem 1** ([A1]). *An extremal character of  $G = T \wr \mathfrak{S}_\infty$  is of Yang–Baxter type if and only if its parameters satisfy: only finitely many  $a_{\zeta, \varepsilon, i}$  are non-zero, they are all rational,  $\mu_\zeta = 0$  for all  $\zeta \in \hat{T}$ , and  $\sum_{\zeta, \varepsilon} \|a_{\zeta, \varepsilon}\| = 1$ .*

The conditions are necessary by restriction to  $\mathfrak{S}_\infty$  and the theorem of [LPW]. For the converse, I construct an explicit pair  $(\pi, R)$  realizing any admissible family of parameters. This work was published in *Algebras and Representation Theory*.

**3.2. A formal reformulation for  $\mathbf{W}_{q,t}(\mathfrak{g})$ .** In the original definition,  $q$  and  $t$  are complex numbers or formal variables, and some steps need care: exponentials of Heisenberg modes, normal ordering, and classical limits. In [A2], we work over the ring  $\mathbb{K} = \mathbb{C}[[h, \beta]]$ , with  $q = e^h$  and  $t = e^{h\beta}$ . In this setting, the limit  $t \rightarrow 1$  is simply the reduction modulo  $\beta$ . We also allow spectral parameters in  $\mathbb{C}^\times q^{\mathbb{Z}} t^{\mathbb{Z}}$ , and not only in  $q^{\mathbb{Z}} t^{\mathbb{Z}}$ . In this way, the limit  $t \rightarrow 1$  contains  $q$ -characters with any spectral parameters. The Heisenberg algebra  $\mathcal{H}_{q,t}(\mathfrak{g})$  acts on Fock modules  $\pi_\mu$  with  $\mu \in \mathbb{C}[\beta] \otimes \mathfrak{h}^*$ , and we complete it for a topology given by the annihilation degree. We restrict  $\mathbf{H}_{q,t}(\mathfrak{g})$  to the space spanned by the normally ordered monomials  $:Y_{i_1}(za_1)^{\pm 1} \cdots Y_{i_m}(za_m)^{\pm 1}:$  (without derivatives).

**Theorem 2** ([A2]).

- (1) *The direct sum of Fock modules  $\bigoplus_\mu \pi_\mu$  is a faithful representation of the completed algebra  $\mathcal{H}_{q,t}(\mathfrak{g})$ .*
- (2) *The normal ordering gives an isomorphism of  $\mathbb{K}$ -vector spaces  $\mathbb{K}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times q^{\mathbb{Z}} t^{\mathbb{Z}}} \simeq \mathbf{H}_{q,t}(\mathfrak{g})$ . In particular, the currents  $Y_i(za)$  are algebraically independent for the normally ordered product.*

**3.3. An algorithm and fundamental fields.** Point (2) of Theorem 2 gives a degree  $d_{i,a}$  for each variable. A monomial  $m$  is *dominant* if  $d_{i,a}(m) \geq 0$  for all  $(i, a)$ , and *generic* if  $d_{i,a}(m) \in \{-1, 0, 1\}$  for all  $(i, a)$ . It is *regular* if, roughly, it does not contain the products

$$Y_i(za) Y_i(zaq^{-2r_i})^{-1}, \quad Y_i(za) Y_i(zat^2)^{-1}, \quad Y_i(za) Y_i(zaq^{-2r_i}t^2)^{-1},$$

and it contains  $Y_i(za)Y_i(zaq^{-2r_i}t^2)$  only together with  $Y_i(zaq^{-2r_i})Y_i(zat^2)$ . A variable  $Y_i(za)$  is *admissible* in  $m$  if  $d_{i,a}(m) = 1$  and  $d_{i,aq^{-2r_i}}(m) = d_{i,at^2}(m) = 0$ .

The algorithm starts from a generic regular dominant monomial  $m$ , with coefficient  $\lambda_m := 1$ . At each step, for each monomial  $M$  already obtained and each admissible variable  $Y_j(za)$  in  $M$ , it creates the monomial  $M' = :M A_j(zaq^{-r_j}t)^{-1}:$ . If  $M = :Y_j(za) \prod_b Y_j(zb) \prod_c Y_j(zc)^{-1} N:$ , where  $N$  does not contain  $Y_j$ , the coefficient of  $M'$  is

$$(1) \quad \lambda_{M'} = \lambda_M \prod_b \frac{(1 - q^{-2r_j} \frac{a}{b})(1 - t^2 \frac{a}{b})}{(1 - \frac{a}{b})(1 - q^{-2r_j} t^2 \frac{a}{b})} \prod_c \frac{(1 - \frac{a}{c})(1 - q^{-2r_j} t^2 \frac{a}{c})}{(1 - q^{-2r_j} \frac{a}{c})(1 - t^2 \frac{a}{c})}.$$

This coefficient is chosen so that the residue of the commutator of  $:M:$  with  $S_j^-(w)$  is cancelled by the term  $:M':$ , thanks to the difference equations. It is a product of functions defined by Kimura and Pestun [KP1]. The algorithm fails if it creates a monomial which is not generic or not regular. We say that it *works* if it does not fail and stops after a finite number of steps.

It is inspired by the Frenkel–Mukhin algorithm, with two differences. First, we treat the admissible variables one by one. Second, the coefficients are not maxima: they are ratios of residues of rational functions in  $q$  and  $t$ . The reason is that  $\mathbf{W}_{q,t}(\mathfrak{g})$  is not a ring, and we cannot describe completely its fields in a fixed  $\mathfrak{sl}_2$ -direction. Similar algorithms were used to compute

$qq$ -characters [BP, FJMV, FJM, K]. In [A2], the spectral parameters are treated in a different way, and this is what we use to prove the next theorem.

**Theorem 3** ([A2]). *Let  $m$  be a generic regular dominant monomial.*

- (a) *Every coefficient (1) is well defined, invertible in  $\mathbb{K}$ , and does not depend on the path. So the algorithm is well defined.*
- (b) *If the algorithm works, then the sum  $T(m) = \sum_M \lambda_M : M : \text{ is in } \mathbf{W}_{q,t}(\mathfrak{g})$ , and  $m$  is its only dominant monomial.*

For (a), we prove that two arrows colored  $i$  and  $j$  going to the same monomial always come from a common monomial, using only the directions  $i, j \in I$ . Then the proof is a computation in rank 2, in types  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$  and  $G_2$ . For (b), we prove that the residues of the commutators with  $S_i^-$  cancel in pairs. For  $S_i^+$ , we use that formula (1) is symmetric under  $q^{-r_i} \leftrightarrow t$ . Figure 1 shows a simple example of the graph of the algorithm.

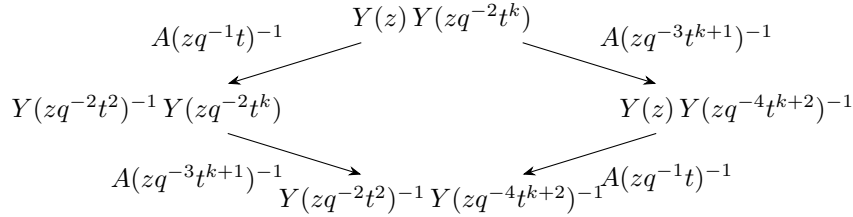


FIGURE 1. The graph of the algorithm for  $\mathfrak{g} = \mathfrak{sl}_2$  and  $m = Y(z)Y(zq^{-2}t^k)$ , with  $k \notin \{0, 2\}$ . In this graph we omit the coefficients. There are two paths to the last monomial, and by Theorem 3(a) they give the same coefficient.

For example, for  $\mathfrak{g} = \mathfrak{sl}_2$  and  $m = Y(z)Y(zq^{-2})$ , the algorithm gives

$$:Y(z)Y(zq^{-2}): + \frac{(q + q^{-1})(qt^{-1} - q^{-1}t)}{q^2t^{-1} - q^{-2}t} :Y(z)Y(zq^{-4}t^2)^{-1}: + :Y(zq^{-2}t^2)^{-1}Y(zq^{-4}t^2)^{-1} :,$$

which is the interpolating function of [FR2, FH2].

**Theorem 4** ([A2]). *Conjecture 1 holds in types  $A_\ell$  ( $\ell \in I$ ),  $B_\ell$  ( $\ell \in I$ ),  $C_\ell$  ( $\ell \in I$ ),  $D_\ell$  ( $\ell \in \{1, \ell - 1, \ell\}$ ),  $E_6$  ( $\ell \in \{1, 5\}$ ),  $E_7$  ( $\ell = 6$ ),  $F_4$  ( $\ell \in \{1, 4\}$ ) and  $G_2$  ( $\ell \in \{1, 2\}$ ).*

So Conjecture 1 is now proved for all nodes in types  $B_\ell$  and  $C_\ell$ . In classical types, we describe a set of monomials indexed by tuples, and we show by induction that the algorithm gives exactly this set, as in [FJM, Theorem 6.1]. In exceptional types, we compute the fields directly. At  $t = 1$ , each field has only one dominant monomial  $Y_{i,1}$ . The specialization  $\mathbf{W}_{q,1}(\mathfrak{g})$  is embedded in the image of the  $q$ -character homomorphism [FR2] (see Section 4.3). Hence, these are the  $q$ -characters of the fundamental representations  $V_{i,a}$ . Moreover, the algorithm works exactly when this representation is thin.

#### 4. CURRENT WORK

**4.1. When does the algorithm work?** Theorem 3 turns the construction of elements of  $\mathbf{W}_{q,t}(\mathfrak{g})$  into a combinatorial question: when does the algorithm work? In [A2], we state two conjectures.

**Conjecture 2** ([A2]). *Let  $m$  be a generic regular dominant monomial. The algorithm works for  $m$  if and only if there is a generic field in  $\mathbf{W}_{q,t}(\mathfrak{g})$ , without derivatives, whose only dominant monomial is  $m$ .*

**Conjecture 3** ([A2]). *Let  $m$  be a generic regular dominant monomial. If the specialization of  $m$  at  $t = 1$  is the dominant monomial of a thin special representation of  $U_q(\widehat{\mathfrak{g}})$ , then the algorithm works for  $m$ .*

Theorem 4 proves Conjecture 3 for fundamental monomials. For the monomial  $Y_2(z)$  in type  $D_4$ , the algorithm fails, because it creates a non-regular monomial. The  $qq$ -character associated to this fundamental representation in type  $D_4$  involves derivatives and is computed by Kimura in [K].

This suggests a link between the multiplicities in the Frenkel-Mukhin algorithm and the success of our algorithm.

More generally, we want to describe the set of dominant monomials for which the algorithm works.

**4.2. Generalization of the algorithm.** The fundamental fields which are not in Theorem 4 correspond to fundamental representations which are not thin. For them, the algorithm must leave the generic and regular setting. Two new situations appear. First, monomials of degree  $d_{i,a} \geq 2$  appear. Their commutators with the screening currents have poles of higher order, and the fields may contain derivatives  $\partial_z Y_i(za)$ , as in the original definition of [FR2]. Second, non-regular monomials appear. Then  $q$ -strings and  $t$ -strings interact, and the coefficients (1) meet the resonance set  $a/b \in \{1, q^{2r_i}, t^{-2}, q^{2r_i} t^{-2}\}$ . This phenomenon occurs in the computation of the fundamental field associated with the middle node in type  $D_4$  in [K]. In this case, our algorithm fails because the corresponding fundamental representation is non-thin and derivatives appear.

A natural question is whether our algorithm extends to all dominant monomials once derivatives are allowed.

**4.3. Limits of the deformed chiral algebra  $\mathbf{W}_{q,t}(\mathfrak{g})$ .** The specializations of  $\mathbf{W}_{q,t}(\mathfrak{g})$  and  $\mathcal{W}_{q,t}(\mathfrak{g})$  are the main reason to study them, and the limits of  $\mathbf{W}_{q,t}(\mathfrak{g})$  are announced in [A2] as the subject of a future paper.

The limit  $t \rightarrow 1$ . Frenkel and Reshetikhin prove that  $\mathbf{W}_{q,1}(\mathfrak{g})$  is embedded in the image of the  $q$ -character homomorphism [FR2], and we reformulate this injection in our formalism in [A2].

By Theorem 4, all fundamental  $q$ -characters lift in types  $A_\ell$ ,  $B_\ell$ ,  $C_\ell$  and  $G_2$ . The next step is to understand products with any spectral parameters. Here, the choice of spectral parameters in  $\mathbb{C}^\times q^{\mathbb{Z}} t^{\mathbb{Z}}$  allows to consider all spectral parameters in  $\mathbb{C}^\times$ .

The limit  $q \rightarrow 1$  with  $t$  fixed. Frenkel and Reshetikhin remark a connection between the limit of the fields in  $\mathbf{W}_{1,t}(\mathfrak{g})$  and the twisted  $t$ -characters of finite dimensional representations of  $U_t \left( {}^L \widehat{\mathfrak{L}}_{\mathfrak{g}} \right)$ .

The limit  $q \rightarrow \epsilon$ . Let  $\epsilon = 1$  if  $\mathfrak{g}$  is simply laced, and  $\epsilon = \exp(i\pi/r^\vee)$  otherwise. In several cases computed by Frenkel and Reshetikhin, the limit  $q \rightarrow \epsilon$  is identified with a twisted  $t$ -character of  $U_t \left( {}^L \widehat{\mathfrak{g}} \right)$ . This gives another motivation to the study of  $\mathbf{W}_{q,t}(\mathfrak{g})$ . The space of fields  $\mathbf{W}_{q,t}(\mathfrak{g})$  would be a bridge between the representation theory of the quantum affine algebra and the quantum Langlands dual affine algebra, as in the folded integrable models of [FHR]. We are working on making this link more precise.

**Question 1.** *Are these specializations onto the whole representation ring? In other words, does every  $q$ -character, respectively every twisted  $t$ -character, lift to  $\mathbf{W}_{q,t}(\mathfrak{g})$ ?*

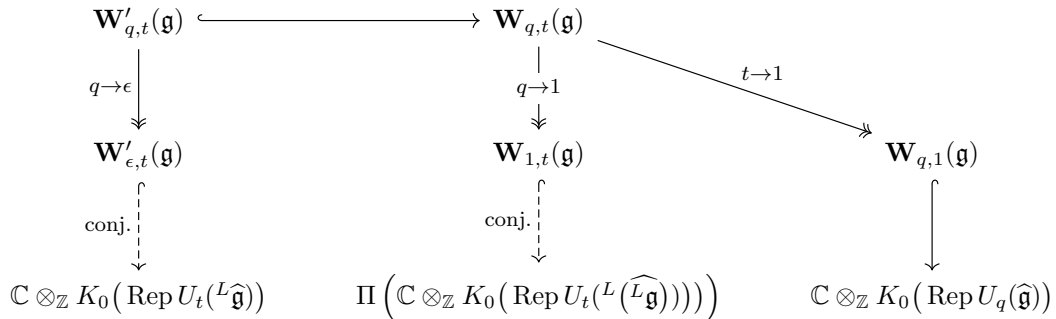


FIGURE 2. The specializations of  $\mathbf{W}_{q,t}(\mathfrak{g})$  and the finite-dimensional representations attached to them. Here  $K_0(\text{Rep } U)$  is the Grothendieck ring of finite-dimensional representations of  $U$ , and  $\Pi$  is a homomorphism. Double-headed arrows are surjective, hooks are injective, and dashed arrows are conjectural.

## 5. FUTURE WORK

**5.1. Classical limits of the deformed  $W$ -algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$ .** As explained in the introduction in [FH1], the study of the specializations of the deformed  $W$ -algebras  $\mathcal{W}_{q,t}(\mathfrak{g})$  is possibly linked with the geometric Langlands correspondence [FH1]. In particular, we want to study two specializations. Let us briefly summarize what is known about those specializations.

- **The limit  $t \rightarrow 1$ .** A completion of  $U_q(\widehat{\mathfrak{g}})$  at the critical level has a large center  $Z_q(\widehat{\mathfrak{g}})$  [RS, DE], and each finite-dimensional module gives central elements through a transfer-matrix construction [RS]. Frenkel and Reshetikhin conjectured that  $Z_q(\widehat{\mathfrak{g}})$  is isomorphic, as a Poisson algebra, to the first classical limit  $\mathcal{W}_{q,1}(\mathfrak{g})$  [FR1, FR2]. Explicit central elements are known in type  $A_\ell$  [FJMR, JLM]. Since Theorem 4 gives all fundamental fields in types  $B_\ell$  and  $C_\ell$ , we want to compute the Poisson brackets of their limits and obtain explicit candidates for generators of  $Z_q(\widehat{\mathfrak{g}})$  even outside type  $A_\ell$ .
- **The limit  $q \rightarrow \epsilon$ .** Frenkel and Reshetikhin consider a subalgebra  $\mathcal{W}'_{q,t}(\mathfrak{g})$  of  $\mathcal{W}_{q,t}(\mathfrak{g})$  such that  $\mathcal{W}'_{\epsilon,t}(\mathfrak{g})$  is again a Poisson algebra, and they conjectured that it is isomorphic to the center of  $U_t({}^L\widehat{\mathfrak{g}})$  at the critical level, where  ${}^L\widehat{\mathfrak{g}}$  is the Langlands dual affine algebra, which is twisted when  $\mathfrak{g}$  is not simply-laced [FR2]. So the same algebra  $\mathcal{W}_{q,t}(\mathfrak{g})$  should see the center for  $\widehat{\mathfrak{g}}$  at one end and the center for  ${}^L\widehat{\mathfrak{g}}$  at the other end.

**Question 2.** *Do the central elements of [FJMR, JLM] generate  $Z_q(\widehat{\mathfrak{g}})$  in type  $A_\ell$ ? In types  $B_\ell$  and  $C_\ell$ , do the Poisson brackets of the limits at  $t = 1$  of the fundamental fields of Theorem 4 give central elements that generate  $Z_q(\widehat{\mathfrak{g}})$ ? And do these same limits generate  $\mathcal{W}_{q,1}(\mathfrak{g})$ ?*

**5.2. The conformal limit.** This is the limit  $h \rightarrow 0$  with  $q = \exp(h)$  and  $t = \exp(h\beta)$ . In this limit,  $\mathcal{W}_{q,t}(\mathfrak{g})$  becomes  $\mathcal{W}_\beta(\mathfrak{g})$  [FR2], and the leading term of the expansion of a field  $T \in \mathbf{W}_{q,t}(\mathfrak{g})$  in powers of  $h$  is an element of  $\mathcal{W}_\beta(\mathfrak{g})$ , as the Virasoro field arises from the generating field of the  $q$ -Virasoro algebra [SKAO]. We want to compute these leading terms for the fundamental fields.

**Question 3.** *Does the expansion in  $h$  of the fundamental fields  $T_i(z)$  give strong generators of  $\mathcal{W}_\beta(\mathfrak{g})$ ?*

**5.3. Crystal structures on the fields.** By Figure 2, each specialization of  $\mathbf{W}_{q,t}(\mathfrak{g})$  embeds into a Grothendieck ring of finite-dimensional representations or into the image of such a ring. Geometric constructions are available on both sides: for simply-laced  $\mathfrak{g}$ , Nakajima obtains the  $q$ -characters of  $U_q(\widehat{\mathfrak{g}})$  from the equivariant K-theory of quiver varieties [N], while Kimura and Pestun construct  $\mathbf{W}_{q,t}(\mathfrak{g})$  itself from the equivariant K-theory of quiver gauge theories, in simply-laced case and in non-simply-laced case through fractional quivers [KP1, KP2]. The coefficients (1) are already products of  $\mathcal{S}$ -functions defined in [KP1]. We want to know how the fields produced by our algorithm sit in this picture.

A first step is combinatorial. Varagnolo and Vasserot equip a set of Laurent monomials, in variables indexed by pairs  $(i, k)$  with  $i \in I$  and  $k \in \mathbb{Z}$ , with a crystal structure of type  $\underline{C}$ , where  $\underline{C}$  is the symmetric Cartan matrix obtained by unfolding the Cartan matrix of  $\mathfrak{g}$  [VV]. There, the operator  $\tilde{f}_i$  multiplies a monomial by the inverse of a root-type variable, and the index of that variable is determined by two statistics of the monomial. Our algorithm also multiplies monomials by inverses of root-type variables, but it applies every admissible variable at each step, whereas  $\tilde{f}_i$  applies exactly one. We want to compare the two constructions.

**Question 4.** *Let  $m$  be a dominant monomial for which the algorithm works. Is there a specialization of the parameters  $q$  and  $t$  such that the set of monomials of the specialized field  $T(m)$  is a subcrystal of the crystal of [VV]?*

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